Solutions to Midterm I.

1. $U = X(x)\, Y(y)$

$X'' + w^2 X = 0 \qquad X = e^{iwx}$ satisfies boundedness [3]

$Y'' - w^2 Y = 0 \qquad Y = \sinh wy$ [4]

satisfies $Y(0) = 0$

$$U = \int_{-\infty}^{\infty} C(w)\, e^{iwx} \sinh wy \, dw \qquad [3]$$

$$U_y(x,b) = f(x) = \begin{cases} 1 & -a < x < a \\ 0 & |x| > a \end{cases}$$

$$U_y(x,y) = \int_{-\infty}^{\infty} C(w)\, e^{iwx}\, w \cosh wy \, dw$$

$$f(x) = \int_{-\infty}^{\infty} C(w)\, e^{iwx}\, w \cosh wb \, dw \qquad [3]$$

$$g(w) = \sqrt{2\pi}\, C(w)\, w \cosh wb = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx}\, dx \quad [5]$$

$$C(w) = \frac{1}{2\pi w \cosh wb} \int_{-a}^{a} 1\, e^{-iwx}\, dx$$

$$C(w) = \frac{1}{2\pi w \cosh wb} \left. \frac{e^{-iwx}}{-iw} \right|_{-a}^{a}$$

$$C(w) = \frac{\sin aw}{4\pi w^2 \cosh(wb)}$$

$$U = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{\sin aw \; e^{iwx} \sinh wy}{w^2 \cosh wb}\, dw$$

$$U = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{\sin aw \; \sinh wy \cos wx}{w^2 \cosh(wb)}\, dw. \leftarrow \qquad [5]$$

2. (a) $$g(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-iwx}\, dx$$ [3]

f is even so

$$g(w) = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} f(x) \cos wx \, dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_1^{2} 1 \cdot \cos wx \, dx$$ [10]

$$g(w) = \frac{2}{\sqrt{2\pi}} \begin{cases} \dfrac{\sin 2w - \sin w}{w} & w \neq 0 \\ 1 & w = 0 \end{cases}$$ [2]

(b).

$$\frac{1}{2}\left[f(x+0) + f(x-0)\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(w)\, e^{iwx}\, dw$$ [2]

Since $g(w)$ is even

$$\frac{1}{2}\left[f(x+0) + f(x-0)\right] = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} g(w) \cos wx \, dw$$

$$= \frac{2}{\sqrt{2\pi}} \frac{2}{\sqrt{2\pi}} \int_0^{\infty} \frac{\sin 2w - \sin w}{w} \cos wx \, dw$$

$$= \frac{2}{\pi} \int_0^{\infty} \frac{\sin w \,(2\cos w - 1)}{w} \cos wx \, dw$$ [4]

At $x = 1$

$$\frac{1}{2} = \frac{2}{\pi} I$$ [2]

$$I = \frac{\pi}{4}$$

3. $\ddot{x}_1 = -6x_1 + 2x_2$ $\qquad x \equiv \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\ddot{x}_2 = -3x_2 + 2x_1$

[7] $\ddot{x} = Ax = \begin{pmatrix} -6 & 2 \\ 2 & -3 \end{pmatrix} x$

$x = e^{i\omega t} v \qquad \dot{v} = 0$

$-\omega^2 v = Av$

$Av = \lambda v \qquad \lambda \equiv -\omega^2$

$\det(A - \lambda I) = \det \begin{pmatrix} -6-\lambda & 2 \\ 2 & -3-\lambda \end{pmatrix} = (\lambda+3)(\lambda+6) - 4 = 0$

[7] $\lambda^2 + 9\lambda + 14 = (\lambda+2)(\lambda+7) = 0 \qquad \lambda = -2, -7.$

$-(6+\lambda)v_1 + 2v_2 = 0$

$2v_1 - (3+\lambda)v_2 = 0$

$\lambda = -2 \qquad 2v_2 = 4v_1 \qquad v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\lambda = -7 \qquad 2v_2 = -v_1 \qquad v = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

so

$\omega_1 = \sqrt{2} \qquad \omega_2 = \sqrt{7} \qquad$ [2] [2]

$x = e^{i\sqrt{2}t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad x = e^{i\sqrt{7}t}\begin{pmatrix} 2 \\ -1 \end{pmatrix} \qquad$ [4] [4]

[8]

4. Let $x_1 \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ & $x_2 \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$. Since $n=4$ and $\exists$ exactly 2 lin ind eigenvectors there must be 2 lin. ind. gen. eigenvectors, call them z_1 & z_2.

Since z_1 satisfies $(A-\lambda I)^2 z = 0$ and x_1 & x_2 do also, $z_1 + \alpha x_1 + \beta x_2$ does also $\forall$ α & $\beta \in R^1$.

So we can choose α & β such that $z_1 = \begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix}$.

Similarly $z_2 = \begin{pmatrix} c \\ d \\ 0 \\ 0 \end{pmatrix}$. Since all these vectors have the same λ eigenvalue any lin. comb. of z_1 & z_2 is also a gen. eigenvector. Clearly $\exists$ ~~the~~ a lin comb

$\tilde{\alpha} z_1 + \tilde{\beta} z_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, and one which is $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

QED.