(g) $a \equiv 3, b \equiv 4 \pmod 5$
$a \equiv 36, b \equiv 42 \pmod{49}$
$a \equiv 7, b \equiv 8 \pmod{11}$
$a \equiv 1, b \equiv 3 \pmod{13}$

(h) $a \equiv 2, b \equiv 4 \pmod 5$
$a \equiv 14, b \equiv 36 \pmod{49}$
$a \equiv 1, b \equiv 10 \pmod{11}$
$a \equiv 11, b \equiv 7 \pmod{13}$

20. Suppose m and n are relatively prime positive integers. Show that there exist integers s and t with $s > 0$ such that $sm + tn = 1$. (Hint: It might be helpful first to consider some specific examples. For instance, can you find s and t, $s > 0$, such that $7s + 22t = 1$?)

The remaining problems are based upon the method for encoding messages described at the end of this section.

21. Suppose $p = 17$, $q = 23$, and $s = 5$. How would you encode each of the following "messages"?

(a) [BB] X
(b) HELP
(c) AIR
(d) BYE
(e) NOW

22. Suppose $p = 5$, $q = 7$, and $s = 5$. Decode each of the following encoded "messages."

(a) [BB] 31
(b) 24
(c) 7
(d) 11
(e) 23

23. Suppose $p = 17$, $q = 59$, and $s = 3$.

(a) If you receive $E = 456$, what is the message?
(b) If you receive $E = 926$, what is the message?

"Discrete Math with Graph Theory", E.G. Goodaire and M.M. Parmenter, Prentice Hall, 1998.

Chapter 4

Induction and Recursion

4.1 Mathematical Induction

One of the most basic methods of proof is proof by *Mathematical Induction*, which is a way to establish the truth of a statement about all the natural numbers or, sometimes, all sufficiently large integers. Mathematical induction is important in every area of mathematics. In addition to the examples presented in this section, other proofs by mathematical induction appear elsewhere in this book in a variety of different contexts. In the index (see *induction*), we draw attention to some theorems whose proofs serve as especially good models of the technique.

Problem 1. A certain store sells envelopes in packages of five and packages of twelve and you want to buy n envelopes. Prove that for every $n \geq 45$, this store can fill an order for exactly n envelopes (assuming an unlimited supply of each type of envelope package).

Solution. If you want to purchase 45 envelopes, you can buy nine packages of five. If you want to buy 46 envelopes, you can buy three packages of twelve and two packages of five. If you want to buy 47 envelopes, you can buy one package of twelve and seven packages of five and if you want 48 envelopes, you would purchase four packages of twelve.

The obvious difficulty with this way of attacking the problem is that it never ends. Even supposing that we continued laboriously to answer the question for n as big as 153, say, could we be sure of a solution for $n = 154$? What is needed is a general, not an ad hoc way to continue; that is, if it is possible to fill an order for exactly k envelopes at this store, we would like to be able to deduce that the store can also fill an order for $k + 1$ envelopes. Then, knowing that we can purchase exactly 45 envelopes and knowing that we can always continue, we could deduce that we can purchase exactly 46 envelopes. Knowing this, and knowing that we can always continue, we would know that we can purchase exactly 47 envelopes. And so on.

Suppose—just suppose—that it is possible to buy exactly k envelopes at this store, where $k \geq 45$. If this purchase requires seven packages of five, then exchanging these for three packages of twelve fills an order of exactly $k+1$ envelopes. On the other hand, if k envelopes are purchased without including seven packages of five, then the order for k envelopes included at most 30 envelopes in packages of five and so, since $k \geq 45$, at least two packages of twelve must have been required. Exchanging these for five packages of five then fills exactly an order for $k+1$ envelopes. We conclude that any order for $n \geq 45$ envelopes can be filled exactly. □

This example demonstrates the key ingredients of a proof by mathematical induction. Asked to prove something about all the integers greater than or equal to a particular given integer—for instance, that any order of $n \geq 45$ envelopes can be filled with packages of five and twelve—we first establish truth for the first integer—for example, $n = 45$—and then show how the truth of the statement for $n = k$ enables us to deduce truth for $n = k+1$.

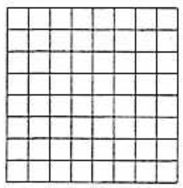
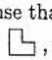


FIGURE 4.1: An 8×8 board.

Problem 2. Chess is a game played on an 8×8 grid, that is, a board consisting of eight rows of eight small squares. (See Fig. 4.1.) Suppose our board is *defective* in the sense that one of its squares is missing. Given a box of L-shaped *trominos* like this, , each of which covers exactly three squares of a chess board, is it possible to tile the board without overlapping or going off the board?

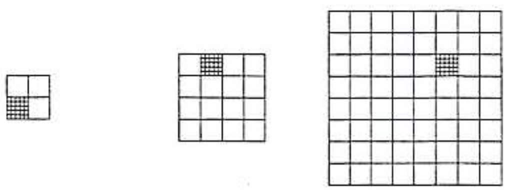


FIGURE 4.2: Three defective boards.

Solution. We begin our solution by thinking of some easier situations. Fig. 4.2 shows defective 2×2, 4×4, and 8×8 boards (with the missing square highlighted in each case). Certainly the 2×2 board can be tiled because its shape is exactly that of a single tromino. A little experimentation would show how to tile the 4×4 board. Rather than proceeding case by case, however, we use the idea suggested by our first example and attempt to understand how a solution for one particular board can be used to obtain a solution for the next bigger board. Suppose then that we know how to tile any $2^k \times 2^k$ defective board. How might we tile a defective board of the next size, $2^{k+1} \times 2^{k+1}$? The idea is to realize that a $2^{k+1} \times 2^{k+1}$ board can be divided into four boards, each of size $2^k \times 2^k$, as shown in Fig. 4.3.

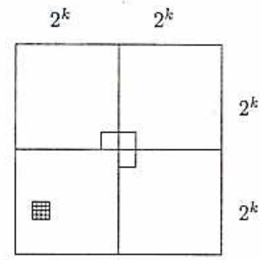


FIGURE 4.3

One of these smaller boards contains the missing square and so is defective. Now place a tromino at the center so as to cover squares in each of the three remaining smaller boards. Each of the boards is now defective and so, by assumption, can be tiled with trominos. So we have tiled the larger board! □

The two examples discussed so far made assertions about infinitely many consecutive integers. In each case, we adopted the following strategy.

- Verify that there is a solution for the smallest integer.
- Show how a solution for one integer leads to a solution for the next.

We now give a formal statement of the principle which has been at work, the *Principle of Mathematical Induction.*

4.1 THE PRINCIPLE OF MATHEMATICAL INDUCTION. Given a statement $\mathcal{P}$ concerning the integer n, suppose

1. $\mathcal{P}$ is true for some particular integer n_0;
2. if $\mathcal{P}$ is true for some particular integer $k \geq n_0$, then it is true for the next integer $k+1$.

Then $\mathcal{P}$ is true for all integers $n \geq n_0$.

In Step 2, the assumption that $\mathcal{P}$ is true for some particular integer is known as the *induction hypothesis.*

In our first example, we had to prove that any order of n envelopes, $n \geq 45$, could be filled with packages of five and of twelve: n_0 was 45 and the induction hypothesis was the assumption that there was a way to purchase k envelopes with packages of five and twelve. In the second example, we had to demonstrate that any defective board of size $2^n \times 2^n$, $n \geq 1$, could be covered in a certain way; n_0 was 1 and the induction hypothesis was the assumption that we could properly cover a $2^k \times 2^k$ board.

Our next example is suggested by the following pattern. Notice that

$$\begin{aligned} 1 &= 1 = 1^2 \\ 1+3 &= 4 = 2^2 \\ 1+3+5 &= 9 = 3^2 \\ 1+3+5+7 &= 16 = 4^2 \\ 1+3+5+7+9 &= 25 = 5^2. \end{aligned}$$

The first odd integer is 1^2; the sum of the first two odd integers is 2^2; the sum of the first three odd integers is 3^2 and so on. It appears as if the sum of the first n odd integers might always be n^2. The picture in Fig. 4.4 adds force to this possibility.

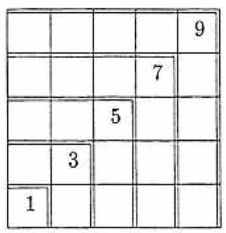


FIGURE 4.4: The sum of the first n odd integers is n^2.

Problem 3. Prove that for any integer $n \geq 1$, the sum of the odd integers from 1 to $2n-1$ is n^2.

Before solving this problem, we remark that the sum in question is often written

$$(1) \qquad 1+3+5+\cdots+(2n-1),$$

where the first three terms here—$1+3+5$—are present just to indicate that odd numbers are being added, beginning with 1, and the last term, $2n-1$, describes the last term and gives a formula for the general term: The second odd number is $2(2)-1$; the third odd number is $2(3)-1$, the kth odd number is $2k-1$. One must not infer from this expression that each of the three numbers 1, 3, 5 is always present. For example, when $n=2$, $2n-1=3$ and so the sum in (1) is, by definition, $1+3$.

One can also describe the sum (1) with *sigma* notation,

$$(2) \qquad 1+3+5+\cdots+(2n-1) = \sum_{i=1}^{n}(2i-1),$$

so named because the capital Greek letter, $\sum$, used to denote summation, is pronounced "sigma." The letter i is called the *index of summation*; the $i=1$ at the bottom and the n at the top mean that the summation starts with $i=1$ and continues with $i=2,3$ and so on, until $i=n$. Thus, the first term in the sum is $2i-1$ with $i=1$; that is, $2(1)-1=1$. The second term is $2i-1$ with $i=2$; that is, $2(2)-1=3$. Summing continues until $i=n$: The last term is $2n-1$.

PAUSE 1. Write $\sum_{i=1}^{4}(2i-1)$ without using sigma notation and evaluate this sum.

Problem 3 asks us to prove that, for all integers $n \geq 1$,

$$(3) \qquad 1+3+5+\cdots+(2n-1) = n^2$$

or, equivalently, that

$$\sum_{i=1}^{n}(2i-1) = n^2.$$

Solution. In this problem, $n_0 = 1$. When $n=1$, $1+3+5+\cdots+(2n-1)$ means "the sum of the odd integers from 1 to $2(1)-1=1$." Thus, the sum is just 1. Since 1^2 is also 1, the statement is true for $n=1$. Now suppose that it is true for some integer $k \geq 1$; in other words, suppose

$$1+3+5+\cdots+(2k-1) = k^2 \qquad \text{induction hypothesis.}$$

We must show that the statement is true for the next integer, $n=k+1$; namely, we must show that

$$1+3+5+\cdots+(2(k+1)-1) = (k+1)^2.$$

Since $2(k+1)-1 = 2k+1$, we have to show

$$1+3+5+\cdots+(2k+1) = (k+1)^2.$$

!!!!—— WAIT A SECOND ——!!!!

The sum on the left is the sum of the odd integers from 1 to $2k+1$; this is the sum of the odd integers from 1 to $2k-1$, plus the next odd integer, $2k+1$:

$$1+3+5+\cdots+(2k+1) = [1+3+5+\cdots+(2k-1)]+(2k+1).$$

Using the induction hypothesis, we have then that

$$\begin{aligned}&1+3+5+\cdots+(2k+1)\\ &\qquad = 1+3+5+\cdots+(2k-1)+(2k+1) = k^2+(2k+1).\end{aligned}$$

Since $k^2+(2k+1)=(k+1)^2$, this is the result we wanted. By the Principle of Mathematical Induction, statement (3) is true for all integers $n \geq 1$. $\square$

Why did we "wait a second" during the above argument? It has been the authors' experience that students sometimes confuse their statement of what is to be proven when $n=k+1$ with the start of their actual proof. Consequently, we strongly recommend the following approach to a proof by mathematical induction:

- verify the statement for $n=n_0$;
- write down the induction hypothesis (the statement for $n=k$) in the form "Now suppose that ... " and be explicit about what is being assumed;
- write down what is to be proven (the statement for $n=k+1$) in the form "We must show that ... " again being very explicit about what is to be shown; and finally (after waiting a second),
- give a convincing argument as to why the statement for $n=k+1$ is true (and make sure this argument uses the induction hypothesis).

We continue with several examples which the student should take as models for solutions by mathematical induction.

Problem 4. Prove that for any natural number $n \geq 1$,

$$1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Solution. When $n=1$, the sum of the integers from 1^2 to 1^2 is 1. Also

$$\frac{1(1+1)(2\cdot 1+1)}{6}=1,$$

so the statement is true for $n=1$. Now suppose that the statement is true for $n=k\geq 1$; that is, suppose that

$$1^2+2^2+3^2+\cdots+k^2=\frac{k(k+1)(2k+1)}{6}.$$

We have to show that the statement is true for $n=k+1$; that is, we have to show that

$$\begin{aligned}&1^2+2^2+3^2+\cdots+(k+1)^2\\ &\qquad = \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6} = \frac{(k+1)(k+2)(2k+3)}{6}.\end{aligned}$$

!!!!—— **WAIT A SECOND** ——!!!!

Observe that

$$\begin{aligned}1^2+2^2+3^2+\cdots+(k+1)^2 &= (1^2+2^2+3^2+\cdots+k^2)+(k+1)^2\\ &= \frac{k(k+1)(2k+1)}{6}+(k+1)^2\\ &= \frac{k(k+1)(2k+1)+6(k+1)^2}{6}\\ &= \frac{(k+1)[k(2k+1)+6(k+1)]}{6}\\ &= \frac{(k+1)[2k^2+7k+6]}{6}\\ &= \frac{(k+1)(k+2)(2k+3)}{6}\end{aligned}$$

which is just what we wanted. By the Principle of Mathematical Induction, the statement is true for all integers $n\geq 1$. $\square$

Problem 5. Prove that for any integer $n\geq 1$, $2^{2n}-1$ is divisible by 3.

Solution. When $n=1$, $2^{2(1)}-1=2^2-1=4-1=3$ is divisible by 3. Now suppose that the statement is true for some integer $k\geq 1$; that is, suppose that $2^{2k}-1$ is divisible by 3. We must prove that the statement is true for $n=k+1$; that is, we must prove that $2^{2(k+1)}-1$ is divisible by 3. The key to what follows is the fact that we must somehow involve the induction hypothesis. Observe that $2^{2(k+1)}-1=2^{2k+2}-1=4(2^{2k})-1$. This is helpful since it introduces 2^{2k}. By the induction hypothesis, $2^{2k}-1=3t$ for some integer t, so $2^{2k}=3t+1$. Now it's smooth sailing. We have

$$\begin{aligned}2^{2(k+1)}-1 &= 2^{2k+2}-1=2^2 2^{2k}-1=4(2^{2k})-1=4(3t+1)-1\\ &= 12t+4-1=12t+3=3(4t+1).\end{aligned}$$

Thus, $2^{2(k+1)}-1$ is divisible by 3, as required. By the Principle of Mathematical Induction, $2^{2n}-1$ is divisible by 3 for all integers $n\geq 1$. $\square$

PAUSE 2. Prove that $3^{2n}-1$ is divisible by 8 for every $n \geq 1$.

Let n be a given positive integer. It is convenient to have some notation for the product of all the integers between 1 and n since this sort of product occurs frequently in statistical and in counting problems. (See Chapter 6.)

4.2 DEFINITION. Define $0! = 1$ and, for any integer $n \geq 1$, define

$$n! = n(n-1)(n-2)\cdots(3)(2)(1).$$

The symbol $n!$ is read "n factorial." The first few factorials are $0! = 1$, $1! = 1$, $2! = 2 \cdot 1 = 2$, $3! = 3(2)(1) = 6$, $4! = 4(3)(2)(1) = 24$. It is useful to notice that $4! = 4(3!)$, $5! = 5(4!)$, and so on. Thus, if one knows $8! = 40{,}320$, then it is easy to deduce that $9! = 9(40{,}320) = 362{,}880$. Factorials grow very quickly. James Stirling (1730) provided an important estimate for the size of $n!$ when n is large.

4.3 STIRLING'S APPROXIMATION.

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}(n/e)^n} = 1; \qquad \text{equivalently,} \quad n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

In these formulae, $e = 2.71828\ldots$ denotes the base of the natural logarithms. Remember from our discussion of the Prime Number Theorem (Theorem 3.34) that one reads "is asymptotic to" at the symbol $\sim$. Thus, Stirling's formula says that $n!$ is asymptotic to $\sqrt{2\pi n}(n/e)^n$. This means that $n!$ is approximately equal to $\sqrt{2\pi n}(n/e)^n$, for large n. For example, $15! \approx \sqrt{30\pi}(15/e)^{15} \approx 1.3 \times 10^{12}$, which, by most people's standards, is indeed a large number.

Our next problem provides another indication of the size of $n!$, albeit a rather crude one. For example, it says that $15! > 2^{15} = 32{,}768$ and $30! > 2^{30} \approx 10^9$.

Problem 6. Prove that $n! > 2^n$ for all $n \geq 4$.

Solution. In this problem, $n_0 = 4$ and certainly $4! = 24 > 16 = 2^4$. Thus, the statement is true for n_0. Now suppose that $k \geq 4$ and the statement is true for $n = k$. Thus, we suppose that $k! > 2^k$. We must prove that the statement is true for $n = k+1$; that is, we must prove that $(k+1)! > 2^{k+1}$. Now

$$(k+1)! = (k+1)k! > (k+1)2^k$$

using the induction hypothesis. Since $k \geq 4$, certainly $k+1 > 2$, so $(k+1)2^k > 2 \cdot 2^k = 2^{k+1}$. We conclude that $(k+1)! > 2^{k+1}$ as desired. By the Principle of Mathematical Induction, we conclude that $n! > 2^n$ for all integers $n \geq 4$. $\square$

PAUSE 3. What was the induction hypothesis in this problem?

PAUSE 4. Why did the induction in this example start at $n = 4$ instead of some smaller integer?

The Principle of Mathematical Induction is one of the most powerful tools of mathematics. With it, one can prove many interesting things, but if it is not applied correctly, one can also prove some interesting things which are not true.

Problem 7. What is the flaw in the following argument which purports to show that

$$2 + 4 + 6 + \cdots + 2n = (n-1)(n+2)$$

for all positive integers n?

"Assume that $2 + 4 + 6 + \cdots + 2k = (k-1)(k+2)$ for some integer k. Then

$$\begin{aligned} 2+4+6+\cdots+2(k+1) &= (2+4+6+\cdots+2k) + 2(k+1) \\ &= (k-1)(k+2) + 2(k+1) \\ &\qquad \text{by the induction hypothesis} \\ &= k^2 + k - 2 + 2k + 2 \\ &= k^2 + 3k \\ &= k(k+3) \\ &= [(k+1)-1)][(k+1)+2] \end{aligned}$$

which is the given statement for $n = k+1$. It follows, by the Principle of Mathematical Induction, that the statement is true for all positive integers n."

Solution. The inductive step, as given, is quite correct, but we neglected to check the case $n = 1$, for which the statement is most definitely false. $\square$

There is another form of the Principle of Mathematical Induction, called the *strong form*, because, at first glance, it appears to be more powerful than the principle stated previously. The two forms are completely equivalent, however: The collection of statements which can be proven true using one form is exactly the collection which can be proven true using the other. It just so happens that in certain problems, the strong form is more convenient than the other.

4.4 PRINCIPLE OF MATHEMATICAL INDUCTION (STRONG FORM). Given a statement $\mathcal{P}$ concerning the integer n, suppose

1. $\mathcal{P}$ is true for some integer n_0; $n_0 - 1$
   $k \geq n_0$
2. if $k > n_0$ is any integer and $\mathcal{P}$ is true for all integers ℓ in the range $n_0 \leq \ell \leq k$, then it is true also for k.
   $k+1$

Then $\mathcal{P}$ is true for all integers $n \geq n_0$.

The two forms of the Principle of Mathematical Induction differ only in the statement of the induction hypothesis (the assumption in the second step). Previously, we assumed the truth of the statement for just one particular integer and we had to prove it true for the next largest integer. In the strong form of mathematical induction, we assume the truth of the statement for *all* integers less than some integer and prove that the statement is true for that integer. When one first encounters mathematical induction, it seems to be the weak form which is always used; problems requiring the strong form are seldom encountered. In the analysis of finite structures, however, the strong form is employed extensively. One wants to acquire knowledge about structures of a certain size from knowledge about similar structures of smaller size. Recall that part of the Fundamental Theorem of Arithmetic states that every natural number greater than 1 is the product of primes. The strong form of mathematical induction affords a very straightforward proof of this result.

Problem 8. Use the strong form of mathematical induction to prove that every natural number $n \geq 2$ is either prime or the product of prime numbers. (See Theorem 3.30.)

Solution. The theorem is a statement about all integers $n \geq 2$. The first such integer, $n_0 = 2$, is prime, so the assertion of the theorem is true. Now let $k > 2$ and suppose that the assertion is true for all positive integers ℓ, $2 \leq \ell < k$; in other words, suppose that every integer ℓ in the interval $2 \leq \ell < k$ is either prime or the product of primes. We must prove that k has this same property. If k is prime, there is nothing more to do. On the other hand, if k is not prime, then k can be factored $k = ab$, where a and b are integers satisfying $2 \leq a, b < k$. By the induction hypothesis, each of a and b is either prime or the product of primes. Thus, k is the product of primes, as required. By the Principle of Mathematical Induction, we conclude that every $n \geq 2$ is prime or the product of primes. ☐

Our next problem demonstrates another common error in "proofs" by mathematical induction.

Problem 9. What is wrong with the following argument which purports to prove that any debt of n dollars, $n \geq 4$ can be repaid with only two-dollar bills?

> Here $n_0 = 4$. We begin by noting that any four-dollar debt can be repaid with two two-dollar bills. Thus, the assertion is true for $n = 4$.
>
> Now let $k > 4$ be any integer and suppose that the assertion is true for all ℓ, $4 \leq \ell < k$. We must prove that the assertion is true for $n = k$. For this, we apply the induction hypothesis to $k - 2$ and see that a $(k-2)$-dollar debt can be repaid with two-dollar bills. Adding one more two-dollar bill allows us to repay k dollars with only two-dollar bills, as required. By the Principle of Mathematical Induction, any debt of $n \geq 4$ dollars can be repaid with two-dollar bills.

Solution. The problem here is that the latter part of the argument does not work if $k = 5$.

The induction hypothesis—that the assertion is true for all ℓ, $4 \leq \ell < k$—was applied to $\ell = k - 2$. If $k = 5$, however, then $k - 2 = 3$, so the induction hypothesis cannot be applied. ☐

We conclude with a brief discussion about the equivalence of the two Principles of Mathematical Induction and the Well Ordering Principle.

Mathematical Induction and Well Ordering

Recall that the Well Ordering Principle (paragraph 3.2) says that any nonempty set of natural numbers has a smallest element. This can be proved using the weak form of the Principle of Mathematical Induction. Here is the argument.

A set containing just one element has a smallest member, the element itself, so the Well Ordering Principle is true for sets of size $n_0 = 1$. Now suppose it is true for sets of size k; that is, assume that any set of k natural numbers has a smallest member. Given a set S of $k + 1$ numbers, remove one element a. The remaining k numbers have a smallest element, say b, and the smaller of a and b is the smallest element of S. This proves that any finite set of natural numbers has a smallest element. We leave to the reader (Exercise 17) the extension of this result to arbitrary subsets of N.

Conversely, one may use the Well Ordering Principle to prove the Principle of Mathematical Induction (weak form). For suppose that $\mathcal{P}$ is a statement involving the integer n which we wish to establish for all integers greater than or equal to some given integer n_0. Assume

1. $\mathcal{P}$ is true for $n = n_0$, and
2. if $\mathcal{P}$ is true for an integer $k \geq n_0$, then it is also true for $k + 1$.

How does the Principle of Well Ordering show that $\mathcal{P}$ is true for all $n \geq n_0$? For convenience we assume that $n_0 \geq 1$. (The case $n_0 < 0$ can be handled with a slight variation of the argument we present below.)

If $\mathcal{P}$ is not true for all $n \geq n_0$, then the set S of natural numbers $n \geq n_0$ for which $\mathcal{P}$ is false is not empty. By the Well Ordering Principle, S has a smallest element a. Now $a \neq n_0$ because we have established that $\mathcal{P}$ is true for $n = n_0$. Thus, $a > n_0$, so $a - 1 \geq n_0$. Also, $a - 1 < a$. By minimality of a, $\mathcal{P}$ is true for $k = a - 1$. By assumption 2, $\mathcal{P}$ is true for $k + 1 = a$, a contradiction. We are forced to conclude that our starting assumption is false: $\mathcal{P}$ must be true for all $n \geq n_0$.

The preceding paragraphs show that the Principles of Well Ordering and Mathematical Induction (weak form) are equivalent. With minor variations in the reasoning above, one can prove that the Principles of Well Ordering and

Mathematical Induction (strong form) are equivalent. It follows, therefore, that the three principles are logically equivalent.

Answers to Pauses

1. $\sum_{i=1}^{4}(2i-1) = [2(1)-1]+[2(2)-1]+[2(3)-1]+[2(4)-1] = 1+3+5+7 = 16$.
2. When $n = 1$, $3^{2(1)} - 1 = 3^2 - 1 = 9 - 1 = 8$ is divisible by 8. Now suppose that the statement is true for some integer $k \geq 1$; that is, suppose that $3^{2k} - 1$ is divisible by 8. Thus, $3^{2k} - 1 = 8t$ for some integer t and so $3^{2k} = 8t + 1$. We have
$$\begin{aligned} 3^{2(k+1)} - 1 = 3^{2k+2} - 1 &= 9(3^{2k}) - 1 \\ &= 9(8t+1) - 1 = 72t + 9 - 1 = 72t + 8 = 8(9t+1). \end{aligned}$$
Thus, $3^{2(k+1)} - 1$ is divisible by 8, as required. By the Principle of Mathematical Induction, $3^{2n} - 1$ is divisible by 8 for all integers $n \geq 1$.
3. The induction hypothesis was that $k! > 2^k$ for some particular integer $k \geq 4$.
4. The statement $n! > 2^n$ is not true for $n < 4$; for example, $3! = 6$, whereas $2^3 = 8$.

EXERCISES

1. Write each of the following sums without using $\sum$ and evaluate.
   (a) [BB][1] $\sum_{i=1}^{5} i^2$
   (b) $\sum_{i=1}^{4} 2^i$
   (c) [BB] $\sum_{t=1}^{1} \sin \pi t$
   (d) $\sum_{j=0}^{2} 3^{j+2}$
   (e) $\sum_{k=-1}^{4}(2k^2 - k + 1)$
   (f) $\sum_{k=0}^{n}(-1)^k$
2. List the elements of each of the following sets:
   (a) $\{\sum_{i=0}^{n}(-1)^i \mid n = 0,1,2,3\}$
   (b) $\{\sum_{i=1}^{n} 2^i \mid n \in \mathsf{N}, 1 \leq n \leq 5\}$

[1] Remember that [BB] means that an answer or solution can be found in the Back of the Book.

3. Prove that it is possible to fill an order for $n \geq 32$ pounds of fish given bottomless wheelbarrows full of five-pound and nine-pound fish.
4. Use mathematical induction to prove the truth of each of the following assertions for all $n \geq 1$.
   (a) [BB] $n^2 + n$ is divisible by 2.
   (b) $n^3 + 2n$ is divisible by 3.
   (c) [BB] $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9.
   (d) $5^n - 1$ is divisible by 4.
   (e) $8^n - 3^n$ is divisible by 5.
   (f) $10^{n+1} + 10^n + 1$ is divisible by 3.
   (g) $a^n - b^n$ is divisible by $a - b$ for any integers a, b with $a - b \neq 0$.
5. (a) [BB] Prove by mathematical induction that
$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$
   for any natural number n.
   (b) Prove by mathematical induction that
$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$
   for any natural number n.
   (c) Use the results of (a) and (b) to establish that
$$(1 + 2 + 3 + \cdots + n)^2 = 1^3 + 2^3 + \cdots + n^3$$
   for all $n \geq 1$.
6. Use mathematical induction to establish the truth of each of the following statements for all $n \geq 1$.
   (a) [BB] $1 + 2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 1$.
   (b) [BB] $1^2 - 2^2 + 3^2 - 4^2 + \cdots + (-1)^{n-1}n^2 = (-1)^{n-1}\frac{n(n+1)}{2}$.
   (c) $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$.
   (d) $1\cdot2\cdot3 + 2\cdot3\cdot4 + 3\cdot4\cdot5 + \cdots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$.
   (e) $\frac{1}{1\cdot2} + \frac{1}{2\cdot3} + \frac{1}{3\cdot4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$.

7. [BB; (a)] Rewrite each of the sums in Exercise 6 using $\sum$ notation.

8. Use mathematical induction to establish each of the formulas below.

(a) [BB] $\sum_{i=1}^{n}(i+1)2^i = n2^{n+1}$

(b) $\displaystyle\sum_{i=1}^{n} \frac{i^2}{(2i-1)(2i+1)} = \frac{n(n+1)}{2(2n+1)}$

(c) $\displaystyle\sum_{i=1}^{n}(2i-1)(2i) = \frac{n(n+1)(4n-1)}{3}$

9. Use mathematical induction to establish each of the following inequalities.

(a) [BB] $2^n > n^2$, for $n \geq 5$.

(b) $(1+\frac{1}{2})^n \geq 1 + \frac{n}{2}$, for $n \in \mathsf{N}$.

(c) For any $p \in \mathsf{R}$, $p > -1$, $(1+p)^n \geq 1 + np$ for all $n \in \mathsf{N}$.

(d) For any integer $n \geq 2$, $\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n} > \frac{13}{24}$.

(e) $\displaystyle\sum_{i=1}^{n} \frac{1}{\sqrt{i}} > \sqrt{n}$ for $n \geq 2$.

10. Suppose $c, x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ are $2n+1$ given numbers. Prove each of the following assertions by mathematical induction.

(a) [BB] $\displaystyle\sum_{i=1}^{n}(x_i + y_i) = \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i$ for $n \geq 1$.

(b) $\displaystyle\sum_{i=1}^{n} cx_i = c\sum_{i=1}^{n} x_i$ for $n \geq 1$.

(c) $\displaystyle\sum_{i=2}^{n}(x_i - x_{i-1}) = x_n - x_1$ for $n \geq 2$.

11. [BB] Find the fault in the following "proof" that in any group of n people, everybody is the same age.

> Suppose $n = 1$. If a group consists of just one person, everybody is the same age. Suppose that in any group of k people, everyone is the same age. Let $G = \{a_1, a_2, \ldots, a_{k+1}\}$ be a group of $k+1$ people. Since each of the groups $\{a_1, a_2, \ldots, a_k\}$ and $\{a_2, a_3, \ldots, a_{k+1}\}$ consists of k people, everybody in each group has the same age, by the induction hypothesis. Since a_2 is in each group, it follows that all $k+1$ people $a_1, a_2, \ldots, a_{k+1}$ have the same age.

12. Find the fault in the following "proof" by mathematical induction that

$$1 + 2 + 3 + \cdots + n = \frac{(2n+1)^2}{8}$$

for all natural numbers n.

> If $1 + 2 + 3 + \cdots + k = \frac{(2k+1)^2}{8}$, then
>
> $$\begin{aligned} 1 + 2 + 3 + \cdots + (k+1) &= (1 + 2 + 3 + \cdots + k) + (k+1) \\ &= \frac{(2k+1)^2}{8} + (k+1) \\ &= \frac{4k^2 + 4k + 1 + 8k + 8}{8} \\ &= \frac{4k^2 + 12k + 9}{8} \\ &= \frac{(2k+3)^2}{8} = \frac{[2(k+1)+1]^2}{8} \end{aligned}$$
>
> and so truth for k implies truth for $k+1$.

13. What is wrong with the following "proof" that any order for $n \geq 10$ pounds of fish can be filled with only five-pound fish?

> We use the strong form of mathematical induction. Here $n_0 = 10$. Since an order for ten pounds of fish can be filled with two five-pound fish, the assertion is true for $n = 10$. Now let $k > 10$ be an integer and suppose that any order for ℓ pounds of fish, $10 \leq \ell < k$, can be filled with only five-pound fish. We must prove that an order for k pounds can be similarly filled. But by the induction hypothesis, we can fill an order for $k - 5$ pounds of fish, so, adding one more five-pounder, we can fill the order for k pounds. By the Principle of Mathematical Induction, we conclude that the assertion is true for all $n \geq 10$.

14. One of several differences between the Canadian and American games of football is that in Canada, a team can score a single point without first having scored a touchdown. So it is quite clear that any score is possible in the Canadian game. Is this so in the American game? Indeed this seems to be the case, even assuming (this is not true!) that in the United States, points can be scored only three at a time (with a field goal) or seven at a time (with a converted touchdown). Here is an argument.

> Assume that k points can be achieved with multiples of 3 or 7. Here's how to reach $k+1$ points. If k points are achieved with at

least two field goals, subtracting these and adding a touchdown gives $k+1$ points. On the other hand, if the k points are achieved with at least two touchdowns, subtracting these and adding five field goals also gives $k+1$ points.

Does this argument show that any score is possible in American football? Can it be used to show something about the nature of possible scores?

15. It is tempting to think that if a statement involving the natural number n is true for many consecutive values of n, it must be true for all n. In this connection, the following example due to Euler is illustrative.
Let $f(n) = n^2 + n + 41$.
   (a) Convince yourself (perhaps with a computer algebra package like Maple or Mathematica) that $f(n)$ is prime for $n = 1, 2, 3, \ldots 39$ but that $f(40)$ is not prime.
   (b) Show that for any n of the form $n = k^2 + 40$, $f(n)$ is not prime.

16. [BB] Prove that a set with n elements, $n \geq 0$, contains 2^n subsets.

17. Suppose that any nonempty finite set of natural numbers has a smallest element. Prove that *any* nonempty set of natural numbers has a smallest element.

18. (a) Prove that for any integer $n \geq 1$, any set of n positive real numbers has a smallest element.
   (b) Prove that the result of (a) is not true for infinite sets of positive real numbers in general but that it is true for some infinite sets.
   (c) What is the name of the principle which asserts that any nonempty set of natural numbers has a smallest element?

19. [BB] Generalize the result of Exercise 26, Section 1.2 by proving that for any $n \geq 1$, any set A and any n sets $B_1, B_2, \ldots, B_n$,
$$A \cup \left(\bigcap_{i=1}^n B_i\right) = \bigcap_{i=1}^n (A \cup B_i).$$

20. For any set A and any n sets $B_1, B_2, \ldots, B_n$, prove that
$$A \cap \left(\bigcup_{i=1}^n B_i\right) = \bigcup_{i=1}^n (A \cap B_i).$$

21. In Section 1.2, we defined the symmetric difference of two sets. More generally, the symmetric difference of $n \geq 3$ sets $A_1, \ldots, A_n$ can be defined inductively as follows:
$$A_1 \oplus \cdots \oplus A_n = (A_1 \oplus \cdots \oplus A_{n-1}) \oplus A_n.$$
Prove that for any $n \geq 2$, $A_1 \oplus A_2 \oplus \cdots \oplus A_n$ consists of those elements in an odd number of the sets $A_1, \ldots, A_n$.

22. Let F_n denote the nth Fermat number; thus, $F_n = 2^{2^n} + 1$. Prove that $F_n - 2$ is the product $F_0 F_1 F_2 \cdots F_{n-1}$ for all $n \geq 1$.

23. Prove Theorem 3.49 (Chinese Remainder) by mathematical induction.

24. [BB] For $n \geq 3$, the greatest common divisor of n nonzero integers a_1, a_2, $\ldots$, a_n can be defined inductively by
$$\gcd(a_1, \ldots, a_n) = \gcd(a_1, \gcd(a_2, \ldots, a_n)).$$
Prove that for all $n \geq 2$, $\gcd(a_1, a_2, \ldots, a_n)$ is an integral *linear combination* of the integers $a_1, a_2, \ldots, a_n$; that is, prove that there exist integers $s_1, \ldots, s_n$ such that $\gcd(a_1, \ldots, a_n) = s_1 a_1 + s_2 a_2 + \cdots + s_n a_n$.

25. The definition of the greatest common divisor of $n \geq 3$ integers given in Exercise 24 differs from that given in the exercises to Section 3.2. Suppose $a_1, \ldots, a_n$ are nonzero integers. Show that $\gcd(a_1, \ldots, a_n)$, as defined in Exercise 24, satisfies the properties given in Definition 3.21.

26. Suppose n and $m_1, m_2, \ldots, m_t$ are natural numbers, and that the m_i are pairwise relatively prime. Suppose each m_i divides n. Prove that the product $m_1 m_2 \cdots m_t$ divides n. (Hint: induction on t and Exercise 10 of Section 3.2.)

27. Define $f\colon \mathsf{Z} \to \mathsf{Z}$ by $f(n) = \begin{cases} n-2 & n \geq 1000 \\ f(f(n+4)) & n < 1000. \end{cases}$
   (a) Find the values of $f(1000-n)$ for $n = 0, 1, 2, 3, 4, 5$.
   (b) Guess a formula for $f(1000-n)$ valid for $n \geq 0$ and prove your answer.
   (c) Find $f(5)$ and $f(20)$.
   (d) What is the range of f?

4.5 **DEFINITION.** A set A of integers is called an *ideal* if and only if
   (i) $0 \in A$,
   (ii) if $a \in A$, then also $-a \in A$, and
   (iii) if $a, b \in A$, then $a + b \in A$.

28. For any integer $n \geq 0$, recall that $n\mathsf{Z} = \{kn \mid k \in \mathsf{Z}\}$ denotes the set of multiples of n.
   (a) Prove that $n\mathsf{Z}$ is an ideal of the integers.
   (b) Let A be any ideal of Z. Prove that $A = n\mathsf{Z}$ for some $n \geq 0$ by establishing each of the following statements.

i. If A contains only one element, then A is of the desired form.

Now assume that A contains more than one element.

ii. Show that A contains a positive number.

iii. Show that A contains a smallest positive number n.

iv. $n\mathbf{Z} \subseteq A$, where n is the integer found in iii.

v. $A \subseteq n\mathbf{Z}$. (Hint: Theorem 3.5, the division algorithm.)

29. [BB] Prove that for every integer $n \geq 2$, the number of lines obtained by joining n distinct points in the plane, no three of which are collinear, is $n(n-1)/2$.

30. An n-sided polygon (commonly shortened to n-gon) is a closed planar figure bounded by n straight sides no two of which intersect unless they are adjacent, in which case they intersect just at a vertex. Thus, a 3-gon is just a triangle, a 4-gon is a quadrilateral, a 5-gon is a pentagon, and so on. An n-gon is *convex* if the line joining any pair of nonadjacent vertices lies entirely within the figure. A rectangle, for example, is convex. Prove that the sum of the interior angles of a convex n-gon is $(n-2)180°$ for all $n \geq 3$.

31. Suppose a rectangle is subdivided into regions by means of straight lines each extending from one border of the rectangle to another. Prove that the regions of the "map" so obtained can be colored with just two colors in such a way that bordering "countries" have different colors.

32. True or false? In each case, give a proof or provide a counterexample which disproves the given statement.

(a) [BB] $5^n + n + 1$ is divisible by 7 for all $n \geq 1$.

(b) $\sum_{k=0}^{n}(k+1) = n(n+3)/2$ for all $n \geq 1$.

(c) If $n \geq 2$, $\gcd\left(\frac{(n+2)!}{3}, \frac{(n+3)!}{2}\right) = \frac{(n+2)!}{6}$.

(d) (For students of calculus) $n^{15} \geq 2^n$ for all $n \geq 1$.

33. (For students who have completed a course in differential calculus) State and prove (by mathematical induction) a formula for $\frac{d}{dx}x^n$ which holds for all $n \geq 1$.

4.6 NOTATION. The product of n elements $a_1, a_2, \ldots, a_n$ is denoted $\prod_{r=1}^{n} a_r$.

34. Prove that for any natural number n and any real number $x \neq \pm 1$,

$$\prod_{r=1}^{n}(1 + x^{2^r}) = \frac{1 - x^{2^{r+1}}}{1 - x^2}.$$

35. [BB] (For students of calculus) The condition $x^2 \neq 1$ is necessary in Exercise 34 since otherwise we would have a denominator of 0 on the right. However,

$$\lim_{x \to 1} \frac{1 - x^{2^{r+1}}}{1 - x^2} \quad \text{and} \quad \lim_{x \to -1} \frac{1 - x^{2^{r+1}}}{1 - x^2}$$

both exist. Use the result of Exercise 34 to find these limits.

36. Find an expression for $\prod_{r=2}^{n} \frac{2r-1}{2r-3}$ valid for $n \geq 2$ and prove by mathematical induction that your answer is correct.

37. **(a)** Prove that the strong form of the Principle of Mathematical Induction implies the Well Ordering Principle.

(b) Prove that the Well Ordering Principle implies the strong form of the Principle of Mathematical Induction. (Assume $n_0 \geq 1$.)

38. In this section, we have studied two formulations of the Principle of Mathematical Induction.

(a) Use either of these to establish the following (peculiar?) third formulation.

Suppose $\mathcal{P}(n)$ is a statement about the natural number n such that
1. $\mathcal{P}(1)$ is true;
2. For any $k \geq 1$, $\mathcal{P}(k)$ true $\implies$ $\mathcal{P}(2k)$ true; and,
3. For any $k \geq 2$, $\mathcal{P}(k)$ true $\implies$ $\mathcal{P}(k-1)$ true.

(b) Prove that for any two nonnegative numbers x and y, $\frac{x+y}{2} \geq \sqrt{xy}$.

(c) Use the Principle of Mathematical Induction in the form given in part (a) to generalize the result of part (b), thus establishing the *arithmetic mean-geometric mean inequality*: For any $n \geq 1$ and any n nonnegative real numbers $a_1, a_2, \ldots a_n$,

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}.$$

4.2 Recursively Defined Sequences

Suppose n is a natural number. How should one define 2^n? One could write

$$2^n = \underbrace{2 \cdot 2 \cdot 2 \cdots 2}_{n\ 2s}$$

but could also write,

(4) $$2^1 = 2 \quad \text{and, for } k \geq 1, \quad 2^{k+1} = 2 \cdot 2^k.$$

The latter statement is an example of a *recursive definition*. It explicitly defines 2^n when $n = 1$ and then, assuming 2^n has been defined for $n = k$, defines it for $n = k+1$. By the Principle of Mathematical Induction, we know that 2^n has been defined for all integers $n \geq 1$.

Another expression that is most naturally defined recursively is $n!$, which was introduced in Section 4.1. If we write

$$0! = 1 \quad \text{and, for } k \geq 0\,, \quad (k+1)! = (k+1)k!,$$

then it follows by the Principle of Mathematical Induction that $n!$ has been defined for every $n \geq 0$.

Sequences of numbers are often defined recursively. A *sequence* is a function whose domain is some infinite set of integers (often N) and whose range is a set of real numbers. Since its domain is countable, we can and usually do describe a sequence by simply listing its range. The sequence which is the function $f\colon \mathsf{N} \to \mathsf{R}$ defined by $f(n) = n^2$, for instance, is generally described by the list $1, 4, 9, 16, \ldots$, the idea being to write down enough numbers from the start of the list that the rest can be inferred. The numbers in the list (the range of the function) are called the *terms* of the sequence. Sometimes we start counting at 0 (if the function has domain $\mathsf{N} \cup \{0\}$) so that the terms are denoted $a_0, a_1, a_2, \ldots$.

The sequence $2, 4, 8, 16, \ldots$ can be defined recursively like this:

(5) $$a_1 = 2 \quad \text{and, for } k \geq 1, \quad a_{k+1} = 2a_k.$$

By this, one understands that $a_1 = 2$ and then, setting $k = 1$ in the second part of the definition, that $a_2 = 2a_1 = 2(2) = 4$. With $k = 2$, $a_3 = 2a_2 = 2(4) = 8$; with $k = 3$, $a_4 = 2a_3 = 2(8) = 16$ and so on. Evidently, (5) defines the sequence we had in mind. Again, the definition is recursive because each term in the sequence beyond the first is defined in terms of the previous term.

The equation $a_{k+1} = 2a_k$ in (5), which defines one member of the sequence in terms of a previous one, is called a *recurrence relation*. The equation $a_1 = 2$ is called an *initial condition*.

There are other possible recursive definitions which describe the same sequence as (5). For example, we could write

$$a_0 = 2 \quad \text{and, for } k \geq 0, \quad a_{k+1} = 2a_k,$$

or we could say

$$a_1 = 2 \quad \text{and, for } k \geq 2, \quad a_k = 2a_{k-1}.$$

The reader should verify that these definitions give the same sequence; namely, $2, 4, 8, 16, \ldots$.

Sometimes, after computing a few terms of a sequence which has been defined recursively, one can guess an explicit formula for a_n. In (5), for instance, $a_n = 2^n$. We say that $a_n = 2^n$ is the *solution* to the recurrence relation. Our goal in this section and the next is to gain some skill at solving recurrence relations.

Problem 10. Write down the first six terms of the sequence defined by $a_1 = 1$, $a_{k+1} = 3a_k + 1$ for $k \geq 1$. Guess a formula for a_n and prove that your formula is correct.

Solution. The first six terms are

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 3a_1 + 1 = 3(1) + 1 = 4 \\ a_3 &= 3a_2 + 1 = 3(4) + 1 = 13 \\ a_4 &= 40 \\ a_5 &= 121 \\ a_6 &= 364. \end{aligned}$$

Since there is multiplication by 3 at each step, we might suspect that 3^n is involved in the answer. After trial and error, we guess that $a_n = \frac{3^n - 1}{2}$ and verify this by mathematical induction.

When $n = 1$, the formula gives $\frac{3^1 - 1}{2} = 1$, which is indeed a_1, the first term in the sequence.

Now assume that $a_k = \frac{3^k - 1}{2}$ for some $k \geq 1$. We wish to prove that $a_{k+1} = \frac{3^{k+1} - 1}{2}$. We have

$$a_{k+1} = 3a_k + 1 = 3\Big(\frac{3^k - 1}{2}\Big) + 1$$

using the induction hypothesis. Hence,

$$a_{k+1} = \frac{3^{k+1}}{2} - \frac{3}{2} + 1 = \frac{3^{k+1} - 1}{2}$$

as required. By the Principle of Mathematical Induction, our guess is correct. ☐

Problem 11. A sequence is defined recursively by $a_0 = 1$, $a_1 = 4$ and $a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 2$. Find the first six terms of this sequence. Then solve the recurrence by guessing a formula for a_n. Establish the validity of your guess.

Solution. Here there are two initial conditions—$a_0 = 1, a_1 = 4$. Also the recurrence relation, $a_n = 4a_{n-1} - 4a_{n-2}$, defines the general term as a function of two previous terms. The first six terms of the sequence are

$$\begin{aligned} a_0 &= 1 \\ a_1 &= 4 \\ a_2 &= 4a_1 - 4a_0 = 4(4) - 4(1) = 12 \\ a_3 &= 4a_2 - 4a_1 = 4(12) - 4(4) = 32 \\ a_4 &= 4a_3 - 4a_2 = 4(32) - 4(12) = 80 \\ a_5 &= 4a_4 - 4a_3 = 4(80) - 4(32) = 192. \end{aligned}$$

Finding a general formula for a_n requires some ingenuity. Let us carefully examine some of the first six terms. We note that $a_3 = 32 = 24 + 8 = 3(8) + 8 = 4(8)$ and $a_4 = 80 = 64 + 16 = 4(16) + 16 = 5(16)$ and $a_5 = 192 = 6(32)$. We are led to guess that $a_n = (n+1)2^n$. To prove this, we use the strong form of mathematical induction (with $n_0 = 0$).

When $n = 0$, we have $(0+1)2^0 = 1(1) = 1$, in agreement with the given value for a_0. When $n = 1$, $(1+1)2^1 = 4 = a_1$. Now that the formula has been verified for $k = 0$ and $k = 1$, we may assume that $k > 1$ and that $a_n = (n+1)2^n$ for all n in the interval $0 \le n < k$. We wish to prove the formula is valid for $n = k$; that is, we wish to prove that $a_k = (k+1)2^k$. Since $k \ge 2$, we know that $a_k = 4a_{k-1} - 4a_{k-2}$. Applying the induction hypothesis to $k-1$ and to $k-2$ (each of which is in the range $0 \le n < k$), we have $a_{k-1} = k2^{k-1}$ and $a_{k-2} = (k-1)2^{k-2}$. Thus,

$$\begin{aligned} a_k &= 4(k2^{k-1}) - 4(k-1)2^{k-2} \\ &= 2k2^k - k2^k + 2^k = k2^k + 2^k = (k+1)2^k \end{aligned}$$

as required. By the Principle of Mathematical Induction, the formula is valid for all $n \ge 0$. □

In effect, our method of verifying the formula $a_n = (n+1)2^n$ in this example amounts simply to checking that it satisfies both initial conditions and also the given recurrence relation. We prefer the more formal approach of mathematical induction since it emphasizes this important concept and avoids pitfalls associated with working on both sides of an equation at once.

As previously mentioned, there is nothing unique about a recursive definition. The sequence in the last example can also be defined by

$$a_0 = 1, a_1 = 4 \quad \text{and, for } n \ge 1, \quad a_{n+1} = 4a_n - 4a_{n-1}.$$

In this case, we again obtain $a_n = (n+1)2^n$. We could also say

$$a_1 = 1, a_2 = 4 \quad \text{and, for } n \ge 1, \quad a_{n+2} = 4a_{n+1} - 4a_n$$

but then, labeling the first term a_1 instead of a_0 would give $a_n = n2^{n-1}$. Other variants are also possible.

Some Special Sequences

Suppose you have \$50 in an old shoe box and acquire a paper route which nets you \$14 a week. Assuming all this money goes into your shoe box on a weekly basis (and you never borrow from it), after your first week delivering papers, your shoe box will contain \$64; after two weeks, \$78; after three weeks, \$92 and so on. A sequence of numbers like $50, 64, 78, 92, \ldots$, where each term is determined by adding the same fixed number to the previous one, is called an *arithmetic sequence*. The fixed number is called the *common difference* of the sequence (because the difference of successive terms is constant throughout the sequence).

Examples 1.

- $50, 64, 78, 92, \ldots$ is an arithmetic sequence with common difference 14.
- $-17, -12, -7, -2, 3, 8, \ldots$ is an arithmetic sequence with common difference 5.
- $103, 99, 95, 91, \ldots$ is an arithmetic sequence with common difference -4.

4.7 DEFINITION. The *arithmetic sequence* with first term a and *common difference* d is the sequence defined by

$$a_1 = a \quad \text{and, for } k \ge 1, \quad a_{k+1} = a_k + d.$$

The general arithmetic sequence thus takes the form

$$a, a+d, a+2d, a+3d, \ldots$$

and it is easy to see that, for $n \ge 1$, the nth term of the sequence is

$$\boxed{a_n = a + (n-1)d.} \tag{6}$$

We leave a formal proof to the exercises and also a proof of the fact that the sum of n terms of the arithmetic sequence with first term a and common difference d is

$$\boxed{S = \tfrac{n}{2}\left[2a + (n-1)d\right].} \tag{7}$$

Example 2. The first 100 terms of the arithmetic sequence which begins $-17, -12, -7, -2, 3, \ldots$ have the sum

$$S = \tfrac{100}{2}[2(-17) + 99(5)] = 50(-34 + 495) = 23{,}050.$$

The 100th term of this sequence is $a_{100} = -17 + 99(5) = 478$ (by (6)). The number 2038 occurs as the 412th term, as we see by solving $-17+(n-1)5 = 2038$.

Many people with paper routes deposit their earnings in a bank account which pays interest instead of into a shoe box which does not. Fifty dollars in a bank account which pays 1% interest per month accumulates to $50+(.01\times 50) = 50(1+.01) = 50(1.01)$ dollars after one month. After another month, the original investment will have accumulated to what it was at the start of the month plus 1% of this amount; that is,

$$50(1.01) + .01(50)(1.01) = 50(1.01)(1 + .01) = 50(1.01)^2.$$

After three months, the accumulation is $50(1.01)^3$ dollars; after twelve months, it is $50(1.01)^{12}$ dollars ($\approx$ \$56.34). A sequence of numbers such as

$$50, 50(1.01), 50(1.01)^2, \ldots$$

in which each term is determined by multiplying the previous term by a fixed number is called a *geometric sequence.* The fixed number is called the *common ratio.*

Examples 3.

- $50, 50(1.01), 50(1.01)^2, \ldots$ is a geometric sequence with common ratio 1.01.
- $3, -6, 12, -24, \ldots$ is a geometric sequence with common ratio -2.
- $9, 3, 1, \frac{1}{3}, \ldots$ is a geometric sequence with common ratio $\frac{1}{3}$.

4.8 **DEFINITION.** The *geometric sequence* with first term a and *common ratio* r is the sequence defined by

$$a_1 = a \quad \text{and, for } k \geq 1, \quad a_{k+1} = ra_k.$$

The general geometric sequence thus has the form

$$a, ar, ar^2, ar^3, ar^4, \ldots$$

the nth term being $a_n = ar^{n-1}$. This is straightforward to prove, as is the following formula for the sum S of n terms, provided $r \neq 1$.

$$\boxed{S = \frac{a(1-r^n)}{1-r}} \qquad (8)$$

Example 4. The sum of 29 terms of the geometric sequence with $a = 8^{12}$ and $r = -1/2$ is

$$S = 8^{12}\,\frac{1-(-\frac{1}{2})^{29}}{1-(-\frac{1}{2})} = 2^{36}\,\frac{1+(\frac{1}{2})^{29}}{\frac{3}{2}} = \frac{2^{36}+2^{7}}{\frac{3}{2}} = \tfrac{1}{3}(2^{37}+2^{8}).$$

PAUSE 1. What is the 30th term of the geometric sequence just described?

Leonardo Fibonacci,[2] also known as Leonardo of Pisa, was one of the brightest mathematicians of the Middle Ages. His writings in arithmetic and algebra were standard authorities for centuries and are largely responsible for the introduction into Europe of the Arabic numerals $0, 1, \ldots, 9$ we use today. Fibonacci was fond of problems, his most famous of which is concerned with rabbits!

Suppose that newborn rabbits start producing offspring by the end of their second month of life and that after this point, they produce a pair a month (one male, one female). Assuming just one pair of rabbits initially, how many pairs of rabbits, Fibonacci asked, will be alive after one year? The sequence which gives the number of pairs at the end of successive months is the famous *Fibonacci sequence.*

After one month, there is still only one pair of rabbits in existence, but after a further month, this pair is joined by its offspring; thus, after two months, there are two pairs of rabbits. At the end of any month, the number of pairs of rabbits is the number alive at the end of the previous month plus the number of pairs alive two months ago, since each pair alive two months ago produced one pair of offspring. We obtain the sequence $1, 1, 2, 3, 5, 8, 13, \ldots$ which is defined recursively as follows.

4.9 THE FIBONACCI SEQUENCE.

$$f_1 = 1, f_2 = 1 \quad \text{and, for } k \geq 2, \quad f_{k+1} = f_k + f_{k-1}.$$

PAUSE 2. Think of the Fibonacci sequence as a function fib: $\mathsf{N} \to \mathsf{N}$. List eight elements of this function as ordered pairs. Is fib a one-to-one function? Is it onto?

PAUSE 3. What is the answer to Fibonacci's question?

Although we have found an explicit formula for the nth term of most of the sequences discussed so far, there are many sequences for which such a formula is difficult or impossible to obtain. (This is one reason why recursive definitions are important.) Is there a specific formula for the nth term of the Fibonacci sequence? As a matter of fact, there is, though it is certainly not one which

[2]Fibonacci was born around the year 1180 and died in 1228. "Fibonacci" is a contraction of "Filius Bonaccii," Latin for "son of Bonaccio."

many people would discover by themselves. We show in the next section (see Problem 14) that the nth term of the Fibonacci sequence is the closest integer to the number

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n. \tag{9}$$

The first few values of $\frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n$ are approximately 0.72361, 1.17082, 1.89443, 3.06525, 4.95967, 8.02492, 12.98460, and 21.00952; the integers closest to these numbers are the first eight terms of the Fibonacci sequence.

Finally, we observe that one must be careful with sequences *apparently* defined recursively, since some recursive definitions do not define actual sequences! Consider, for example,

$$a_1 = 1 \quad \text{and, for } k > 1, \quad a_k = \begin{cases} 1 + a_{k/2} & \text{if } k \text{ is even} \\ 1 + a_{3k-1} & \text{if } k \text{ is odd.} \end{cases}$$

What happens if we try to write down the first few terms of this sequence?

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 1 + a_1 = 1 + 1 = 2 \\ a_3 &= 1 + a_8 = 1 + (1 + a_4) = 2 + a_4 = 2 + (1 + a_2) = 3 + a_2 = 5 \\ a_4 &= 1 + a_2 = 1 + 2 = 3, \end{aligned}$$

but then
$$\begin{aligned} a_5 &= 1 + a_{14} = 1 + (1 + a_7) = 2 + a_7 \\ &= 2 + (1 + a_{20}) = 3 + a_{20} = 3 + (1 + a_{10}) \\ &= 4 + a_{10} = 4 + (1 + a_5) = 5 + a_5 \end{aligned}$$
and, to our dismay, we have reached the absurdity $5 = 0$. Obviously, no sequence has been defined.

PAUSE 4. Compute the first six terms of the sequence "defined" as follows:

$$a_1 = 1 \quad \text{and, for } k > 1, \quad a_k = \begin{cases} 1 + a_{k/2} & \text{if } k \text{ is even} \\ 1 + a_{3k+1} & \text{if } k \text{ is odd.} \end{cases}$$

Answers to Pauses

1. Here $n = 30$, $a = 8^{12} = (2^3)^{12} = 2^{36}$ and $r = -(2^{-1})$, so
$$a_{30} = 2^{36}[-(2^{-1})]^{29} = -2^{36}2^{-29} = -2^7 = -128.$$

2. The most obvious eight pairs are $(1,1)$, $(2,1)$, $(3,2)$, $(4,3)$, $(5,5)$, $(6,8)$, $(7,13)$ and $(8,21)$. The Fibonacci function is not one-to-one because $(1,1) \in$ fib and $(2,1) \in$ fib but $1 \neq 2$. It's not onto since, for example, 4 is not in the range.

3. The number of rabbits in existence after twelve months is the *thirteenth* term of the Fibonacci sequence, 233.

4. $a_1 = 1$; $a_2 = 1 + a_1 = 1 + 1 = 2$; $a_3 = 1 + a_{10} = 1 + 1 + a_5 = 2 + 1 + a_{16} = 3 + a_{16}$. Now $a_{16} = 1 + a_8 = 1 + 1 + a_4 = 1 + 1 + 1 + a_2 = 3 + 2 = 5$, so that $a_3 = 3 + 5 = 8$. Continuing, $a_4 = 1 + a_2 = 3$; $a_5 = 1 + a_{16} = 1 + 5 = 6$; $a_6 = 1 + a_3 = 1 + 8 = 9$. The first six terms are $1, 2, 8, 3, 6, 9$.

EXERCISES

1. Give recursive definitions of each of the following sequences:
   (a) [BB] $1, 5, 5^2, 5^3, 5^4, \ldots$
   (b) $5, 3, 1, -1, -3, \ldots$
   (c) $4, 1, 3, -2, 5, -7, 12, -19, 31, \ldots$
   (d) $1, 2, 0, 3, -1, 4, -2, \ldots$

2. (a) [BB] Find the first seven terms of the sequence $\{a_n\}$ defined by
$$a_1 = 16, \text{ and for } k \geq 1,\ a_{k+1} = \begin{cases} 1 & \text{if } a_k = 1 \\ a_k/2 & \text{if } a_k \text{ is even} \\ (a_k - 1)/2 & \text{if } a_k \neq 1 \text{ is odd.} \end{cases}$$
   (b) Repeat part (a) with $a_1 = 17$.
   (c) Repeat part (a) with $a_1 = 18$.
   (d) Repeat part (a) with $a_1 = 100$.

3. Let $a_1, a_2, a_3, \ldots$ be the sequence defined by $a_1 = 1$, $a_{k+1} = 3a_k$ for $k \geq 1$. Prove that $a_n = 3^{n-1}$ for all $n \geq 1$.

4. [BB] Suppose $a_1, a_2, a_3, \ldots$ is a sequence of integers such that $a_1 = 0$ and, for $n > 1$, $a_n = n^3 + a_{n-1}$. Prove that for every integer $n \geq 1$,
$$a_n = \frac{(n-1)(n+2)(n^2+n+2)}{4}.$$

5. Define the sequence $a_1, a_2, a_3, \ldots$ by $a_1 = 0$, $a_2 = \frac{1}{2}$ and, for $k \geq 1$, $a_{k+2} = \frac{1}{2}(a_k + a_{k+1})$. Find the first seven terms of this sequence. Prove that for every $n \geq 1$,
$$a_n = \tfrac{1}{3}(1 - (-1/2)^{n-1}).$$

6. [BB] Let $a_1, a_2, a_3, \ldots$ be the sequence defined by $a_1 = 1$ and, for $n > 1$, $a_n = 2a_{n-1} + 1$. Write down the first six terms of this sequence. Guess a formula for a_n and prove that your guess is correct.

7. Let $a_1, a_2, a_3, \ldots$ be the sequence defined by $a_1 = \frac{3}{2}$ and $a_n = 5a_{n-1} - 1$ for $n \geq 2$. Write down the first six terms of this sequence. Guess a formula for a_n and prove that your guess is correct.

8. Suppose $a_0, a_1, a_2, \ldots$ is a sequence such that $a_0 = a_1 = 1$ and for $n \geq 1$, $a_{n+1} = n(a_n + a_{n-1})$.

   (a) Find a_2, a_3, a_4 and a_5.

   (b) Guess a formula for a_n, valid for $n \geq 0$, and use mathematical induction to prove that your guess is correct.

9. [BB] Consider the sequence defined by $a_1 = 1$, $a_{n+1} = (n+1)^2 - a_n$ for $n \geq 1$. Find the first six terms. Guess a general formula for a_n and prove that your answer is correct using mathematical induction.

10. Let $a_1, a_2, a_3, \ldots$ be the sequence defined by $a_1 = 1$, $a_{k+1} = (k+1)a_k$ for $k \geq 1$. Find a formula for a_n and prove that your formula is correct.

11. [BB] Suppose $a_1, a_2, a_3, \ldots$ is a sequence of integers such that $a_1 = 0$, $a_2 = 1$ and for $n > 2$, $a_n = 4a_{n-2}$. Guess and then establish by mathematical induction a formula for a_n.

12. A sequence is defined recursively by $a_0 = 2$, $a_1 = 3$ and $a_n = 3a_{n-1} - 2a_{n-2}$ for $n \geq 2$.

    (a) Find the first five terms of this sequence.

    (b) Guess a formula for a_n.

    (c) Verify that your guess in (b) is correct.

    (d) Find a formula for a_n which involves only one preceding term.

13. Let $a_1, a_2, a_3, \ldots$ be the sequence defined by $a_1 = 1$, $a_2 = 0$ and for $n > 2$, $a_n = 4a_{n-1} - 4a_{n-2}$. Prove that $a_n = 2^n(1 - \frac{n}{2})$ for all $n \geq 1$.

14. Let $a_1, a_2, a_3, \ldots$ be the sequence defined by

$$a_1 = 1, \quad \text{and for } k \geq 1, \quad a_{k+1} = k^2 a_k.$$

    Find the first six terms of this sequence. Guess a general formula for a_n and prove your answer by mathematical induction.

15. Consider the arithmetic sequence with first term 2 and common difference 3.

    (a) [BB] Find the first ten terms and the 123rd term of this sequence.

    (b) [BB] Does 752 belong to this sequence? If so, what is the number of the term where it appears?

    (c) Repeat (b) for 1023 and 4127.

    (d) [BB] Find the sum of the first 75 terms of this sequence.

16. Consider the arithmetic sequence with first term 7 and common difference $-1/2$.

    (a) Find the 17th and 92nd terms.

    (b) Find the sum of the first 38 terms.

17. Consider the arithmetic sequence which begins $5, 9, 13$.

    (a) Find the 32nd and 100th terms of this sequence.

    (b) Does 125 belong to the sequence? If so, where does it occur?

    (c) Repeat (b) for the numbers 429 and 1000.

    (d) Find the sum of the first 18 terms.

18. Consider the arithmetic sequence which begins $-123, -117$.

    (a) Find the 77th and 121st terms of this sequence.

    (b) For each of the numbers $-65, 0, 1773$ determine whether or not the number is in the sequence and if it is, its term number.

    (c) Find the sum of the first 77 terms of the sequence.

19. An arithmetic sequence begins $116, 109, 102$.

    (a) Find the 300th term of this sequence.

    (b) Determine whether or not -480 belongs to this sequence. If it does, what is its term number?

    (c) Find the sum of the first 300 terms of the sequence.

20. Establish formulas (6) [BB] and (7) for the nth term and the sum of the first n terms of the arithmetic sequence with first term a and common difference d.

21. [BB] Consider the geometric sequence with first term 59,049 and common ratio $-\frac{1}{3}$.

    (a) Find the first ten terms and the 33rd term of this sequence.

    (b) Find the sum of the first 12 terms.

22. Consider the geometric sequence which begins $-3072, 1536, -768$.

    (a) Find the 13th and 20th terms of this sequence.
    (b) Find the sum of the first nine terms.

23. If the first term of a geometric sequence is 48 and the sixth term is $-\frac{3}{2}$, find the sum of the first ten terms.

24. (a) Find, to four decimal places, the 129th term of the geometric sequence which begins $-0.00001240, 0.00001364$.
    (b) Find the approximate sum of the first 129 terms of the sequence in (a).

25. [BB] Verify formula (8) for the sum of n terms of a geometric sequence with first term a and common ratio $r \neq 1$.

26. Consider the sequence defined recursively by $a_1 = 1$ and for $n > 1$, $a_n = \sum_{i=1}^{n-1} a_i$. Write down the first six terms of this sequence, guess a formula for a_n valid for $n \geq 2$, and prove your answer.

27. (a) Find the sum of 18 terms of the geometric sequence with first term 7/1024 and common ratio 8.
    (b) [BB] Suppose $|r| < 1$. Explain why the sum of the first n terms of the geometric sequence with first term a and common ratio r is approximately $\frac{a}{1-r}$.
    (c) [BB] Find, approximately, $\sum_{k=0}^{100} 3/2^k$.
    (d) Find the approximate sum of the first 1 million terms of the geometric sequence which begins $144, 48, 16$.

28. (a) Find the 19th and 100th terms of the geometric sequence which has first term 98,415 and common ratio $\frac{1}{3}$.
    (b) Find the sum of the first 15 terms of the sequence in (a).
    (c) Find the approximate sum of the first 10,000 terms of the sequence in (a).

29. Given that each sum below is the sum of part of an arithmetic or geometric sequence, find each sum.

    (a) [BB] $75 + 71 + 67 + 63 + \cdots + (-61)$
    (b) $75 + 15 + 3 + \frac{3}{5} + \cdots + \frac{3}{5^7}$
    (c) $-52 - 41 - 30 - 19 + \cdots + 949$
    (d) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots + \frac{1}{2^{50}}$
    (e) $2 + 6 + 18 + 54 + \cdots + 354{,}294$

30. A bank account pays interest at the rate of $100i\%$ a year. Assume an initial balance of P, which accumulates to s_n after n years.

    (a) Find a recursive definition for s_n.
    (b) Find a formula for s_n.

31. Maurice borrows \$1000 at an interest rate of 15%, compounded annually.

    (a) How much does Maurice owe after two years?
    (b) In how many years will the debt grow to \$2000?

32. On January 1, 1993, you have \$50 in a savings account, which pays interest at the rate of 1% per month. At the end of January and at the end of each month thereafter, you deposit \$56 to this account. Assuming no withdrawals, what will be your balance on January 1, 1994?

33. On June 1, you win \$1 million in a lottery and immediately acquire numerous "friends," one of whom offers you the deal of a lifetime. In return for the million, she'll pay you a cent today, two cents tomorrow, four cents the next day, eight cents the next, and so on, stopping with the last payment on June 21.

    (a) Assuming you take this deal, how much money will you receive on June 21?
    (b) Should you take the deal? Explain.
    (c) Would you take the deal if payments continued for the entire month of June?

34. Define a sequence $\{a_n\}$ recursively as follows:

$$a_0 = 0, \qquad \text{and for } n > 0, \qquad a_n = a_{\lfloor n/5 \rfloor} + a_{\lfloor 3n/5 \rfloor} + n.$$

    Prove that $a_n \leq 20n$ for all $n \geq 0$. (Recall that $\lfloor x \rfloor$ denotes the floor of the real number x. See paragraph 2.6.)

35. Suppose we think of the Fibonacci sequence as going backward as well as forward. What seven terms precede $1, 1, 2, 3, 5, 8, \ldots$? How is f_{-n} related to f_n?

36. [BB] Let $\{f_n\}$ denote the Fibonacci sequence. Prove that $f_{n+1}f_n = \sum_{i=1}^{n} f_i^2$ for all $n \geq 1$.

37. Represent the Fibonacci sequence by $f_1 = f_2 = 1$, $f_n = f_{n-1} + f_{n-2}$ for $n > 2$.

    (a) Verify the formula $f_1 + f_2 + f_3 + \cdots + f_n = f_{n+2} - 1$ for $n = 4, 5, 6$.
    (b) Prove that the formula in (a) is valid for all $n \geq 1$.

38. Show that, for $n \geq 2$, the nth term of the Fibonacci sequence is less than $(7/4)^{n-1}$. (Use the definition of the Fibonacci sequence, not the approximation to f_n given in equation 9).

39. [BB] What is wrong with the following argument, which purports to prove that all the Fibonacci numbers after the first two are even?

> Let f_n denote the nth term of the Fibonacci sequence. We prove that f_n is even for all $n \geq 3$ using the strong form of the Principle of Mathematical Induction. The Fibonacci sequence begins 1, 1, 2. Certainly $f_3 = 2$ is even and so the assertion is true for $n_0 = 3$. Now let k be an integer $k > 3$ and assume that the assertion is true for all n, $2 \leq n < k$; that is, assume that f_n is even for all $n < k$. We wish to show that the assertion is true for $n = k$; we wish to show that f_k is even. But $f_k = f_{k-1} + f_{k-2}$. Applying the induction hypothesis to $k-1$ and to $k-2$, we conclude that each of f_{k-1} and f_{k-2} is even, hence, so is the sum. By the Principle of Mathematical Induction, f_n is even for all $n \geq 3$.

40. Let $f_1 = f_2 = 1$, $f_k = f_{k-1} + f_{k-2}$ for $k > 2$ be the Fibonacci sequence. Which terms of this sequence are even? Prove your answer.

41. For $n \geq 1$, let a_n denote the number of ways to express n as the sum of natural numbers, taking order into account. For example, $3 = 3 = 1+1+1 = 2+1 = 1+2$, so $a_3 = 4$.
   (a) [BB] Find the first five terms of the sequence $\{a_n\}$.
   (b) Guess and then establish a formula for a_n.

42. For $n \geq 1$, let b_n denote the number of ways to express n as the sum of 1s and 2s, taking order into account. Thus, $b_4 = 5$ because $4 = 1+1+1+1 = 2+2 = 2+1+1 = 1+1+2 = 1+2+1$.
   (a) Find the first five terms of the sequence $\{b_n\}$.
   (b) Find a recursive definition for b_n and identify this sequence.

43. (a) [BB] Let a_n be the number of ways of forming a line of n people distinguished only by sex. For example, there are four possible lines of two people—MM,MW,WM,WW—so $a_2 = 4$. Find a recurrence relation satisfied by a_n and identify the sequence $a_1, a_2, a_3, \ldots$
   (b) Let a_n be the number of ways in which a line of n people can be formed such that no two males are standing beside each other. For example, $a_3 = 5$ because there are five ways to form lines of three people with no two males beside each other; namely, FFF, MFF, FMF, FFM, MFM. Find a recurrence relation satisfied by a_n and identify the sequence $a_1, a_2, a_3, \ldots$

44. Define the Fibonacci sequence by $f_1 = f_2 = 1$, $f_{n+1} = f_n + f_{n-1}$ for $n \geq 2$.
   (a) Prove that $\gcd(f_{n+1}, f_n) = 1$ for all $n \geq 1$.
   (b) Prove that $f_n = f_{n-m+1}f_m + f_{n-m}f_{m-1}$ for any positive integers n and m with $n > m > 1$.
   (c) Prove that for any positive integers n and m, the greatest common divisor of f_n and f_m is $f_{\gcd(n,m)}$.

45. Suppose u_n and v_n are sequences defined recursively by

$$u_1 = 0, v_1 = 1, \text{ and, for } n \geq 1,$$
$$u_{n+1} = \tfrac{1}{2}(u_n + v_n), v_{n+1} = \tfrac{1}{4}(u_n + 3v_n).$$

   (a) Prove that $v_n - u_n = 1/4^{n-1}$ for $n \geq 1$.
   (b) Prove that u_n is an increasing sequence; that is, $u_{n+1} > u_n$ for all $n \geq 1$.
   (c) Prove that v_n is a decreasing sequence; that is, $v_{n+1} < v_n$ for all $n \geq 1$.
   (d) Prove that $u_n = \frac{2}{3} - \frac{1}{6} \cdot \frac{1}{4^{n-2}}$ for all $n \geq 1$.

   (This problem is taken from a Portuguese examination designed to test the level of mathematical knowledge of graduating high school students. It was reprinted in *Focus*, the newsletter of the Mathematical Association of America, **13**, no. 3, June 1993, p. 13.)

4.10 DEFINITION. If $f\colon A \to A$ is a function, its powers f^n are defined recursively by

$$f^1 = f \quad \text{and, for } n > 1, \quad f^n = f \circ f^{n-1}.$$

46. Suppose $f\colon \mathsf{N} \to \mathsf{N}$ is defined by

$$f(m) = \begin{cases} 2m/3 & \text{if } m \equiv 0 \pmod 3 \\ (4m-1)/3 & \text{if } m \equiv 1 \pmod 3 \\ (4m+1)/3 & \text{if } m \equiv 2 \pmod 3. \end{cases}$$

   (a) Prove that f is one-to-one and onto.
   (b) [BB] Find the first ten terms of the sequence $f^n(1)$.
   (c) Find the sequence $f^n(2)$, $n \geq 1$.
   (d) Find the sequence $f^n(4)$, $n \geq 1$.
   (e) Find the first ten terms of the sequence $f^n(8)$, $n \geq 1$.[3]

[3] It is unknown whether or not the terms of this sequence ever repeat.

47. Define $g\colon \mathsf{N} \to \mathsf{N}$ by

$$g(m) = \begin{cases} m/2 & \text{if } m \text{ is even} \\ (3m+1)/2 & \text{if } m \text{ is odd} \end{cases}.$$

This function is known as the "$3m+1$" function. It is suspected that for *any* starting number m, the sequence $g(m), g^2(m), g^3(m), \ldots$, eventually terminates with 1. Verify this assertion for each of the five integers $m =$ 341, 96, 104, 336, and 133.[4]

48. (For students who have had a course in linear algebra) Give a recursive definition of the *determinant* of an $n \times n$ matrix, for $n \geq 1$.

4.3 Solving Recurrence Relations; The Characteristic Polynomial

Recursively defined sequences were introduced in the previous section. Given a particular recurrence relation and certain initial conditions, the reader was encouraged to guess a formula for the nth term and prove that a guess was correct. Guessing is an important tool in mathematics and a skill which can be sharpened through practice, but we now confess that there is a definite procedure for solving most of the recurrence relations we have encountered so far.

In this section, we describe a procedure for solving recurrence relations of the form

(10) $$a_n = ra_{n-1} + sa_{n-2} + f(n)$$

where r and s are constants and $f(n)$ is some function of n. Such a recurrence relation is called a *second-order linear recurrence relation with constant coefficients*. If $f(n) = 0$, the relation is called *homogeneous*. *Second-order* refers to the fact that the recurrence relation (10) defines a_n as a function of the two terms preceding it. The reader should consult more specialized books in combinatorics for a general treatment of constant coefficient recurrence relations where a_n is a function of any number of terms of the form ca_{n-i}, $c \in \mathsf{R}$.[5]

[4] While this conjecture has been established for all integers $m < 2^{40} \approx 10^{12}$, it is unknown whether it holds for all integers! This problem has attracted the interest of many people, some of whom have offered a sizeable monetary reward for its solution! We refer the interested reader to the excellent article, "The $3x+1$ Problem and Its Generalizations," by Jeffrey C. Lagarias, *American Mathematical Monthly*, **92** (1985) no. 1, 1–23.

[5] See, for example, Alan Tucker, *Applied Combinatorics*, Wiley (1980).

Examples 5. Here are some second-order linear recurrence relations with constant coefficients.

- $a_n = a_{n-1} + a_{n-2}$, the recurrence relation which appears in the definition of the Fibonacci sequence. This is homogeneous with $r = s = 1$. Notice that we have modified slightly the recurrence relation $a_{n+1} = a_n + a_{n-1}$ defined in (4.9) so that it is readily seen to be of the type we are considering here.
- $a_n = 5a_{n-1} - 6a_{n-2} + n$. Here $r = 5$, $s = -6$, $f(n) = n$.
- $a_n = 3a_{n-1}$. This is homogeneous with $r = 3$, $s = 0$.

Examples 6. Consider the following two recurrence relations.

- $a_n = 5a_{n-1} - 3a_{n-3}$
- $a_n = a_{n-1}a_{n-2} + n^2$

Neither is of interest to us in this section. The first is not second-order while the second is not linear.

With the homogeneous recurrence relation $a_n = ra_{n-1} + sa_{n-2}$, which can be rewritten in the form

$$a_n - ra_{n-1} - sa_{n-2} = 0,$$

we associate the quadratic polynomial

$$x^2 - rx - s,$$

which is called the *characteristic polynomial* of the recurrence relation. Its roots are called the *characteristic roots* of the recurrence relation. For example, the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$ has characteristic polynomial $x^2 - 5x + 6$ and characteristic roots 2 and 3.

The following theorem, whose proof is left to the exercises, shows how to solve any second-order *homogeneous* recurrence relation with constant coefficients.

4.11 THEOREM. Let x_1 and x_2 be the roots of the polynomial $x^2 - rx - s$. Then the solution of the recurrence relation $a_n = ra_{n-1} + sa_{n-2}$, $n \geq 2$, is

$$\begin{cases} a_n = c_1x_1^n + c_2x_2^n & \text{if } x_1 \neq x_2 \\ a_n = c_1x^n + c_2nx^n & \text{if } x_1 = x_2 = x. \end{cases}$$

In each case, c_1 and c_2 are constants determined by initial conditions.