MATH 205 L01 W 2005
MIDTERM SOLUTIONS

1. For each of the following answer True or False.

   (a) False, the Goldbach Conjecture is still unproved.
   (b) False, Gauss' portrait is on the German 10 Mark bill.
   (c) True
   (d) False, it is not prime.
   (e) True
   (f) True
   (g) False, Hamilton lived in the 19th century.
   (h) False, there is also Centre Recherche Montreal and PIMS.
   (i) True
   (j) False, the converse is the Four Colour Theorem and is true but not the given statment.

2. (a) $5,836,017 = 3 \times 11 \times 176849$

   (b) $5,836,039 = 11 \times 530549$

   (c) $4 + 503,216,043_{(7)}$ is divisible by 7 since in base 7 it ends in a 0.

   (d) $2^{39} - 1 = (2^3)^{13} - 1 = 8^{13} - 1 = (8-1)(8^{12} + 8^{11} + ... + 1)$, so it has 7 as a factor (similarly it has 8191=$2^{13} - 1$ as a factor).

   (e) 1,547=$7 \times 13 \times 17$

3. Using mathematical induction, prove $\sum\limits_{j=1}^{n} j^2 = n(n+1)(2n+1)/6$.

   Proof: Let $\mathcal{P}_n$ be the above statement.

   (a) $\mathcal{P}_1 : LHS = 1^2 = 1, \;\; RHS = 1(2)(3)/6 = 1, \;\; LHS = RHS.$

   (b) $\mathcal{P}_n \Rightarrow \mathcal{P}_{n+1}$: Assuming the inductive hypothesis $\mathcal{P}_n$,

$$1^2+2^2+...+n^2+(n+1)^2 = \frac{n(n+1)(2n+1)}{6}+(n+1)^2 = (n+1)[\frac{n(2n+1)}{6}+n+1].$$

   Simplifying the RHS gives

$$(n+1)\frac{2n^2+n+6n+6}{6} = (n+1)\frac{2n^2+7n+6}{6} = \frac{(n+1)(n+2)(2n+3)}{6},$$

which proves $\mathcal{P}_{n+1}$. □

4. Give the names of three famous twentieth century mathematicians. Here are a few that were mentioned in class: Hilbert, Einstein, Coxeter, Wiles, von Neumann, Haken, Appel, Matiyasevic, Fields. In addition the name of any Fields medalist would be fine, and the entire list was distributed in class.

5. Consider the sequence $F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3,$ $F_5 = 5, \ldots$.

   (a) Write out the next 7 terms of this sequence. $F_6 = 8, F_7 = 13, F_8 = 21, F_9 = 34, F_{10} = 55, F_{11} = 89, F_{12} = 144.$

   (b) This famous sequence is named the Fibonacci sequence.

   (c) Using inductive reasoning and the values $F_0, \ldots, F_{12}$ as above, make a plausible statement about when F_n is divisible by 4, i.e. for which values of n will F_n be divisible by 4.

   Solution: Note that F_0, F_6, F_{12} are the only ones divisible by 4 on the list. A reasonable conclusion is that F_n is divisible by 4 iff n is divisible by 6.

   (d) Show that $F_n + F_{n+3} = 2F_{n+2}$, for example (for $n = 2$) $1 + 5 = 2 \times 3$.

   [Hint : You may assume the basic defining relation of this sequence $F_{n+1} = F_n + F_{n-1}$. Show then that $F_{n+2} = 2F_n + F_{n-1}$, derive a similar formula for F_{n+3}, and use these to prove the theorem.]

   Several proofs are possible, here is one.

   $F_{n+1} = F_n + F_{n-1}$,
   $F_{n+2} = F_{n+1} + F_n = 2F_n + F_{n-1}$,
   $F_{n+3} = F_{n+2} + F_{n+1} = 3F_n + 2F_{n-1}$.

   Thus $F_n + F_{n+3} = 4F_n + 2F_{n-1} = 2F_{n+2}$. □

6. (a) Convert the following Mayan number to base 10:

   It equals 5396.

   (b) Convert the base 10 number 17,357 to a Mayan number.

   (c) Convert the base 10 number 17,357 to base 5.

   $1023412_{(5)}$

7. Solve the equation $34x + 10y = 1000$, where x, y are positive integers.

   Using Euclidean algorithm and the Z-process one gets

$$(-2) \times 34 + 7 \times 10 = 2$$

$$(-1000) \times 34 + 3500 \times 10 = 1000$$

$$(-1000 + 10s) \times 34 + (3500 - 34s) \times 10 = 1000$$

   Take s = 101, then $x = 20$, $y = 32$. A couple of other solutions are possible, $x = 10$ and $y = 66$, also $x = 25$ and $y = 15$.

8. Complete the addition and multiplication tables below for base 5 arithmetic. Then carry out the operations, all in base 5, of
   $31241 - 13042$, 234×32.

+	1	2	3	4
1	2	3	4	10
2	3	4	10	11
3	4	10	11	12
4	10	11	12	13

×	2	3	4
2	4	11	13
3	11	14	22
4	13	22	31

   $31241 - 13042 = 13144$, $234 \times 32 = 14143$