Mathematical Induction

Two Proofs

1. Theorem : If $\mathcal{G}$ is a finite tree, then $V_{\mathcal{G}} - E_{\mathcal{G}} = 1$.

   Proof by mathematical induction: We use induction on $V_{\mathcal{G}}$. Let $\mathcal{P}_n$ be the statement that for any tree $\mathcal{G}$ with $V_{\mathcal{G}} = n$, $V_{\mathcal{G}} - E_{\mathcal{G}} = 1$, $n \geq 1$.

   (a) $\mathcal{P}_1$: If $n = 1$ then the graph $\mathcal{G}$ consists of just a single vertex with no edges. In that case $V_{\mathcal{G}} - E_{\mathcal{G}} = 1 - 0 = 1$.

   (b) $\mathcal{P}_n \Rightarrow \mathcal{P}_{n+1}$. To prove this let $\mathcal{G}$ have $n + 1$ vertices. Since this graph is a finite tree there must be at least one vertex a with a single neighbour b (because any path in a tree cannot double back to a previous vertex, so must ultimately end since the tree is finite). Form a new graph $\mathcal{H}$ by deleting the vertex a and deleting all of the edge ab except for keeping b. Then $\mathcal{H}$ is still connected so also a tree, and it has one less vertex as well as one less edge compared to $\mathcal{G}$.

   By $\mathcal{P}_n$, one has $V_{\mathcal{H}} - E_{\mathcal{H}} = 1$. It follows that

   $$V_{\mathcal{G}} - E_{\mathcal{G}} = (V_{\mathcal{H}} + 1) - (E_{\mathcal{H}} + 1) = V_{\mathcal{H}} - E_{\mathcal{H}} + 1 - 1 = 1 + 1 - 1 = 1 \ .$$

   This proves $\mathcal{P}_{n+1}$. □

2. Theorem : The sum of a geometric progression is given by

   $$a + ar + ar^2 + \ldots + ar^n = a \cdot \frac{r^{n+1} - 1}{r - 1} \ .$$

   Proof by mathematical induction : Let $\mathcal{P}_n$ be the above formula, for $n \geq 0$.

   (a) $\mathcal{P}_0$: The LHS (left hand side) equals a. The RHS equals $a \cdot \dfrac{r^1 - 1}{r - 1} = a \cdot \dfrac{r - 1}{r - 1} = a = \text{LHS}$.

   (b) $\mathcal{P}_n \Rightarrow \mathcal{P}_{n+1}$.

   $a + ar + ar^2 + \ldots + ar^n + ar^{n+1} = (a + ar + ar^2 + \ldots + ar^n) + ar^{n+1}$ (associative law)

   $= a \cdot \dfrac{r^{n+1} - 1}{r - 1} + ar^{n+1} = a \cdot \left(\dfrac{r^{n+1} - 1}{r - 1} + r^{n+1} \right)$ (by $\mathcal{P}_n$)

   $= a \cdot \left(\dfrac{r^{n+1} - 1 + r^{n+2} - r^{n+1}}{r - 1} \right)$ (common denominator)

   $= a \cdot \dfrac{r^{n+2} - 1}{r - 1}$. (simplification)

   This proves $\mathcal{P}_{n+1}$, thus completing the mathematical induction. □