MATH 205 L01 W 2006
MIDTERM AND SOLUTIONS

1. For each of the following answer True or False. Do not write T or F. [20]

   (a) Fermat is famous for his proof of the infinitude of primes. False

   (b) The 10 Euro bill has a picture of Gauss. False

   (c) The prime number $2^{24,036,583} - 1$ is a Mersenne prime. True

   (d) Euler showed that $2^{2^5} + 1$ is prime. False

   (e) The binary number system uses only two symbols. True

   (f) Any tree can be coloured using only two colours. True

   (g) Sophie Germain was one of the most famous mathematicians of the 20th century. False

   (h) There is only one mathematics institute in Canada, the Pacific Institute of Mathematical Sciences . False

   (i) Any connected graph with all vertices of even degree admits an Euler circuit. True

   (j) Any graph that can be coloured with 4 or fewer colours is planar. False

2. Each of the following numbers is composite. For each one, find a factor (you do not have to factor the number completely). [15]

   (a) 8,806,047 3 is a factor (use the 3 test)

   (b) 9,876,757 11 is a factor (use the 11 test)

   (c) $107^{33} + 5$ 2 is a factor (it is the sum of two odd numbers)

   (d) $2^{33} - 1$ $2^{33} - 1 = (2^3)^{11} - 1 = 8^{11} - 1 = (8-1)(8^{10} + 8^9 + \ldots + 8 + 1)$ so 7 is a factor

   (e) 1,589 7 is a factor (a little trial and error)

3. Describe the advantages of a positional system to denote integers, as compared to grouping systems or other possible methods. Also mention at least one example of a grouping system, and at least one example of a positional system. [10]

   The main advantage of a positional system is its efficiency, and the ease with which the basic arithmetical operations can be done. It only uses a finite number of symbols, for example our decimal system uses only ten symbols, the Mayan system only 3. And large numbers can be handled without lengthy expressions. Also, each number can be written in only one way, for example in the Roman system IV=IIII can be written two ways.

   Examples of grouping systems: Tally, Egyptian, ancient Chinese, Roman

   Examples of positional system: Decimal (Hindu-Arabic), binary, Mayan, modern Chinese

   Remark: The Babylonian system can be considered as an early example of a positional system

4. For each of the mathematicians below complete the table. The choice of nations should be taken from the list : Canada, France, Germany, Greece, Hungary, Ireland, Italy, Russia, Sweden, Switzerland, USA. The choice of centuries should be taken from the list : Antiquity (over 1,500 years ago), 15th, 16th, 17th, 18th, 19th, 20th. Note that the 15th century means the years 1400-1499, etc. [10]

	Nation	Century
Hamilton	Ireland	19th
Pythagoras	Greece	Antiquity
Coxeter	Canada	20th
Fermat	France	17th
Euler	Switzerland	18th

   Remark : Germany or Russia also acceptable for Euler, since he worked in both countries.

5. Consider the sequence $F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3,$ $F_5 = 5, ...$. [10]

(a) Write out the next 7 terms of this sequence.

$$F_6 = 8, F_7 = 13, F_8 = 21, F_9 = 34$$
$$F_{10} = 55, F_{11} = 89, F_{12} = 144.$$

(b) This famous sequence is named the Fibonacci sequence.

(c) Using inductive reasoning and the values $F_0, ..., F_{12}$ as above, make a plausible statement about when F_n is divisible by 5, i.e. for which values of n will F_n be divisible by 5.

Since $F_0, F_5, \text{and} F_{10}$ are divisible by 5, it is reasonable to conjecture that F_n is di8visible by 5 iff n is divisible by 5.

(d) Show that $2F_{n+2} = F_n + F_{n+3},$ for example (for $n = 2$) $2 \times 3 = 1 + 5$.

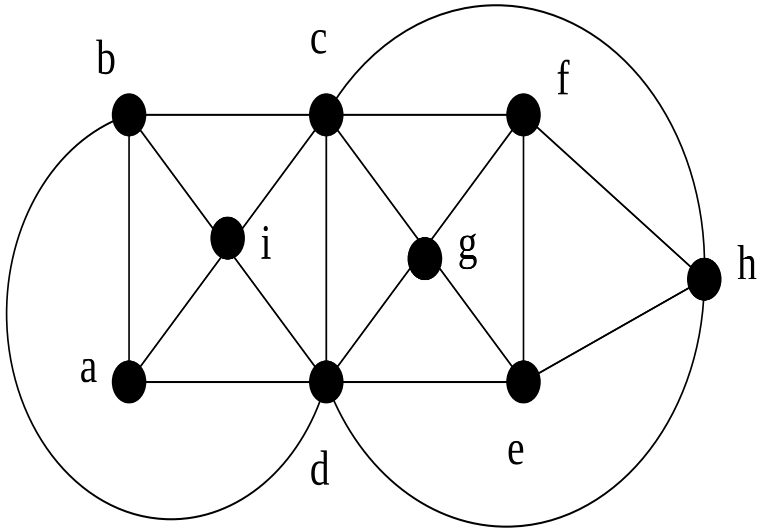
[Hint : You may assume the basic defining relation of this sequence $F_{n+1} = F_n + F_{n-1}$. Show then that $F_{n+2} = 2F_n + F_{n-1},$ derive a similar formula for F_{n+3}, and use these to prove the theorem.]

Proof: $F_{n+2} = F_{n+1} + F_n = F_n + F_{n-1} + F_n = 2F_n + F_{n-1},$
which proves the first formula.

Then $F_{n+3} = F_{n+2} + F_{n+1} = (2F_n + F_{n-1}) + (F_n + F_{n-1}) = 3F_n + 2F_{n-1},$ let's call this the second formula. Adding F_n to the second formula gives

$$F_n + F_{n+3} = 4F_n + 2F_{n-1} = 2(2F_n + F_{n-1}) = 2F_{n+2} \ ,$$

as required to prove (where we used the first formula in the last step).



6. In the graph G shown above, indicate (by giving the order of the vertices)

(a) A Hamilton cycle ______________

Many answers are possible, one such is $abcfhegdia$

(b) An Euler path or circuit ______________

No Euler circuit is possible, but many Euler paths are possible - beginning at a and ending at d (or vica-versa). One such is $abcibdaidcfhcgfehdegd$

(c) Colour the second copy of G (below) using four colours, say A, B, C, D. [10]

Many ways to four colour the graph are possible, make sure you are colouring the *vertices*.

7. Solve the equation $17x+39y = 1000$, where x, y are positive integers. [15]

Answer : $x = 29$, $y = 13$

8. (a) Write the Roman numeral MMDCCCXLIV in base 10.

2,844

(b) Write the Egyptian numeral in base 10.

21,324

(c) Convert 671 (base 10) to base 5. [10]

$10141_{(5)}$