Practice Problems S3

1. Determine whether the following matrices are elementary matrices or not; write down the inverses of the elementary matrices (explain your answer):

   (a) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, (b) $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, (c) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, (d) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

   (e) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$, (f) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$.

2. Find an invertible matrix U such that the product $R = UA$ is the reduced row-echelon form of A if

$$A = \begin{bmatrix} 1 & -1 & 3 & 5 \\ 3 & -2 & 1 & -2 \\ -1 & 1 & 1 & 3 \end{bmatrix}.$$

3. Express the following matrix as a product of elementary matrices:

$$A = \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}.$$

4. Find the matrix of the reflection in the line $y = -x$.

5. Find a rotation or a reflection that is equal to

   (a) reflection in the y-axis followed by rotation through $\pi/2$;

   (b) rotation through $\pi/2$ followed by reflection in the line $y = x$.

6. Given $T([1\ -2]^T) = [3\quad 4]^T$ and $T([-2\quad 5]^T) = [-1\quad 4]^T$, find $T([-4\quad 3]^T)$ if T is a linear transformation.

7. Consider a Markov chain that starts in state 1 with transition matrix $P = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$.

   (a) Explain why the chain is regular.

   (b) Find the probability that the chain is in state 1 after 2 transitions.

   (c) Find the steady-state vector for the chain.

Recommended Problems:

Pages 68 - 69: 1; 2a, b; 3a; 5a, b; 6 a,b; 7; 8b, c;
Pages 80-81: 1. b, c; 2. a; 3, 4, 5, 9, 10, 12; Pages 101-102: 1, 2, a, c;
Page 101-102: 1. a, b, f, g, h, k, l, m, n, o, p; 5. a, b; 6, 7, 8, 9, 11, 13, 14, 15;

Solutions

1. Exercise.

2. To find such an invertible matrix U that transforms A into its reduced row-echelon form R, we have to bring $[A|I_3]$ to $[R|U]$:

$$[A|I_3] = \left[\begin{array}{cccc|ccc} \mathbf{1} & -1 & 3 & 5 & 1 & 0 & 0 \\ 3 & -2 & 1 & -2 & 0 & 1 & 0 \\ -1 & 1 & 1 & 3 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|ccc} \mathbf{1} & -1 & 3 & 5 & 1 & 0 & 0 \\ 0 & \mathbf{1} & -8 & -17 & -3 & 1 & 0 \\ 0 & 0 & 4 & 8 & 1 & 0 & 1 \end{array}\right]$$

(Row 2 has been replaced by row2-3xrow1, and Row 3 by $r3 + r1$)

$$r3 \text{ by } \frac{r3}{4} \longrightarrow \left[\begin{array}{cccc|ccc} \mathbf{1} & -1 & 3 & 5 & 1 & 0 & 0 \\ 0 & \mathbf{1} & -8 & -17 & -3 & 1 & 0 \\ 0 & 0 & \mathbf{1} & 2 & 1/4 & 0 & 1/4 \end{array}\right]$$

$$r1 \text{ by } r1{-}3(r3) \text{ and } r2 \text{ by } r2{+}8(r3) \longrightarrow \left[\begin{array}{cccc|ccc} \mathbf{1} & -1 & 0 & -1 & 1/4 & 0 & -3/4 \\ 0 & \mathbf{1} & 0 & -1 & -1 & 1 & 2 \\ 0 & 0 & \mathbf{1} & 2 & 1/4 & 0 & 1/4 \end{array}\right]$$

$$r1 \text{ by } r1 + r2 \longrightarrow \left[\begin{array}{cccc|ccc} \mathbf{1} & 0 & 0 & -2 & -3/4 & 1 & 5/4 \\ 0 & \mathbf{1} & 0 & -1 & -1 & 1 & 2 \\ 0 & 0 & \mathbf{1} & 2 & 1/4 & 0 & 1/4 \end{array}\right].$$

So, $R = \begin{bmatrix} \mathbf{1} & 0 & 0 & -2 \\ 0 & \mathbf{1} & 0 & -1 \\ 0 & 0 & \mathbf{1} & 2 \end{bmatrix}$ and $U = \begin{bmatrix} -3/4 & 1 & 5/4 \\ -1 & 1 & 2 \\ 1/4 & 0 & 1/4 \end{bmatrix}$. One can check that $R = UA$. Note that this invertible matrix U is not unique. It depends on the sequence of row operations performed to bring A to its reduced row-echelon form.

3. The matrix $A = \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}$ has determinant $\det(A) = 5 \times 1 - 2 \times 3 = -1 \neq 0$. So, A is invertinble. It can be carried to the identity matrix I_2 by row operations.

   Step 1: Substract 2 times row 2 from row 1:

$$A = \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \quad \left(I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow F_1 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \right)$$

Step 2: Substract 2 times row 1 from row 2:

$$\longrightarrow \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \quad \left(I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow F_2 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \right)$$

Step 3: Multiply row 2 by -1:

$$\longrightarrow \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \left(I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow F_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right)$$

Step 4: Substract row 2 from row 1:

$$\longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \left(I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow F_4 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \right)$$

We have $A^{-1} = F_4F_3F_2F_1$. Therefore, $A = E_1E_2E_3E_4$, where E_1, E_2, E_3 and E_4 are elementary matrices given by $E_1 = F_1^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $E_2 = F_2^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, $E_3 = F_3^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $E_4 = F_4^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

4. Denote by $\vec{v}'$ the reflection of $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ in the line $y = x$ and $\vec{v}''$ the reflection of $\vec{v}'$ in the line $y = -x$. We know that $\vec{v}' = \begin{bmatrix} y \\ x \end{bmatrix}$. From the figure, we see that $\vec{v}''$ is the reflection of $\vec{v}'$ about the origin. Therefore, $\vec{v}'' = \begin{bmatrix} -y \\ -x \end{bmatrix}$. It follows that the reflection about the line $y = -x$ is a linear transformation with matrix $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$. It is the composition of the reflection in the line $y = x$ with the reflection about the origin: $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$.

Figure 1: Reflection of vectors of $\mathbb{R}^2$ about the line $y = -x$:

5. (a) The reflection of vectors of $\mathbb{R}^2$ in the y-axis is defined by $T([x\ \ y]^T) = [-x\ \ y]^T$ and the rotation $R_{\pi/2}$ is given by $R_{\pi/2}([x\ \ y]^T) = [-y\ \ x]$. We have: $R_{\pi/2} \circ T([x\ \ y]^T) = R_{\pi/2}(T([x\ \ y]^T)) = R_{\pi/2}([-x\ \ y]^T) = [-y\ \ -x]^T$. Therefore, the reflection in the y-axis followed by the rotation through $\pi/2$ is the reflection about the line $y = -x$.

   (b) Let $R_{\pi/2}([x\ \ y]^T)$ be the rotation through $\pi/2$ and $T([x\ \ y]^T) = [y\ \ x]^T$ the reflection about the line $y = x$. We have $T \circ R_{\pi/2}([x\ \ y]^T) = T(R_{\pi/2}([x\ \ y]^T)) = T([-y\ \ x]^T) = [x\ \ -y]^T$. It is the reflection in the x-axis.

6. Given that $T([1\ \ -2]^T) = [3\ \ 4]^T$ and $T([-2\ \ 5]^T) = [-1\ \ 4]^T$, $T([-4\ \ 3]^T)$ can be computed if we can express $[-4\ \ 3]^T$ as a linear combination of vectors $[1\ \ -2]^T$ and $[-2\ \ 5]^T$, i.e.; find a and b such that $[-4\ \ 3]^T = a[1\ \ -2]^T + b[-2\ \ 5]^T$. We have:

$$\begin{cases} -4 &= a - 2b \\ 3 &= -2a + 5b \end{cases} \implies \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

$$\implies \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}^{-1} \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

$$= \frac{1}{5-4} \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -4 \\ 3 \end{bmatrix} = \begin{bmatrix} -14 \\ -5 \end{bmatrix},$$

i.e., $a = -14$ and -5. It follows that

$$T([-4\ \ 3]^T) = aT([1\ \ -2]^T) + bT([-2\ \ 5]^T) = -14[3\ \ 4]^T - 5[-1\ \ 4]^T = [-37\ \ 76]^T.$$

7. This Markov chain with transition matrix $P = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$, starting in state 1, has initial state vector $S_0 = [1\ \ 0]^T$.

   (a) Since $P^1 = P$ is the transition matrix with only nonzero entries, all powers of P also have nonzero entries. So, at least one power of P has nonzero entries, i.e., the chain is regular.

   (b) $S_{m+1} = PS_m = P^{m+1}S_0$, where $S_0 = [1\ \ 0]^T$. So, $S_1 = PS_0 = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$ and the state-vector S_2 after two transitions is $S_2 = PS_1 = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \mathbf{\frac{5}{9}} \\ \frac{4}{9} \end{bmatrix}$. So, the probability for the chain to be in state 1 after 2 transitions is $\mathbf{\frac{5}{9}}$.

(c) Since the chain is regular, it has a steady state-vector which is the only probability vector S solving the homogeneous system $(I_2 - P)X = 0$.

$$I_2 - P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

($r_1 \to 3 \times r_1/2$ and $r_2 \to r_1 + r_2$). So, the homogeneous system $(I_2 - P)X = 0$ has general solution $X = t[1 \ \ 1]^T$, $t \in \mathbb{R}$, In particular the steady state-vector (which is a solution to the homo. syst.) has the form $S = t[1 \ \ 1]$, for some $t \in \mathbb{R}$. Since S is a probability vector, the sum of its entries is 1, therefore $t = 1/2$ and $S = \frac{1}{2}[1 \ \ 1]^T$.