Practice Problems S4

1. By inspection, find the determinants of the following matrices:

   (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$; (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; (c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$;

   (d) $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 4 \\ 2 & -4 & 6 \end{bmatrix}$; (e) $\begin{bmatrix} 1 & 0 & 4 & 9 \\ -8 & -7 & 12 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 5 & 3 \end{bmatrix}$.

2. Compute the determinants of the following matrices

   (a) $A = \begin{bmatrix} -2 & 1 & 3 \\ 1 & -7 & 4 \\ -2 & 1 & 3 \end{bmatrix}$; (b) $A = \begin{bmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{bmatrix}$.

3. Find the inverse of $A = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$ using the adjoint formula.

4. Given $A = \begin{bmatrix} 3 & -1 & 2 \\ 5 & 5 & -2 \\ 1 & 2 & 3 \end{bmatrix}$, find the $(1,3)$-entry of A^{-1}.

5. If A and B are 4×4-matrices with $\det(A) = -2$ and $\det(B) = 2$, find

   (a) $\det\left(\text{adj}(A)B^T A^4 (B^2)^{-1}\right)$; (b) $\det\left(A^3 (B^2)^T \left((\text{adj}(A))^{-1}\right)^3 B^{-1}\right)$.

6. Let $A = \begin{bmatrix} a & b & c \\ 1 & -1 & 2 \\ d & e & f \end{bmatrix}$, $B = \begin{bmatrix} a & b & c \\ 3 & -2 & 1 \\ d & e & f \end{bmatrix}$ and $C = \begin{bmatrix} a & b & c \\ 1 & 0 & -3 \\ d & e & f \end{bmatrix}$

   be 3×3-matrices. If $\det(A) = 4$ and $\det(B) = 5$, find $\det(C)$.

7. For which values of $c \in \mathbb{R}$ is $A = \begin{bmatrix} 1 & c & 0 \\ 2 & 0 & c \\ c & -1 & 1 \end{bmatrix}$ invertible?

8. Solve the following system by Cramer's rule:

   (a) $\begin{cases} x + 2y = 4 \\ 3x + 7y = 13 \end{cases}$; (b) $\begin{cases} 3x - 2y + 4z = -3 \\ 5x + 3y + z = 0 \\ 2x + 6y - 5z = 6 \end{cases}$.

Recommended Problems:

Pages 114-115: 1. a, b, f, g, h, k, l, m, n, o, p; 5. a, b; 6, 7, 8, 9, 11, 13, 14, 15;
Pages 126-127: 1. a, c; 2. a, b, c, d; 3, 4, 6, 8, 9, 10.

<div style="text-align: center;">Solutions</div>

1. (a): -1; (b): -3; (c)1; (d): 0; (e):-21

2. (a): 0; (b): $\det(A) = \begin{vmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{vmatrix} = - \begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 1 & 8 & 0 \end{vmatrix} =$

$$- \begin{vmatrix} 0 & 9 & 3 \\ 0 & 3 & 3 \\ 1 & 8 & 0 \end{vmatrix} = - \begin{vmatrix} 9 & 3 \\ 3 & 3 \end{vmatrix} = -18.$$

3. $A^{-1} = \dfrac{1}{\det(A)} \operatorname{adj}(A)$, where $\det(A) = -1$ and $\operatorname{adj}(A) = [c_{ij}(A)]^T =$

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & -1 \\ -1 & 1 & -1 \end{bmatrix}^T = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}. \text{ Therefore, } A^{-1} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}.$$

4. A has determinant $\det(A) = 84.$ The $(\mathbf{1}, \mathbf{3})$-entry of A^{-1} is $\frac{1}{\det(A)} c_{\mathbf{31}}(A) =$

$$\tfrac{1}{84}(-1)^{3+1} \begin{vmatrix} -1 & 2 \\ 5 & -2 \end{vmatrix} = -\tfrac{8}{84} = -\tfrac{2}{21}.$$

5. If an $n \times n$ matrix A is invertible, then $\det\big(\operatorname{adj}(A)\big) = \big(\det(A)\big)^{n-1}$.
   For $n = 4$, $\det(A) = -2$ and $\det(B) = 2$, we have:

   (a)

$$\begin{aligned} \det\big(\operatorname{adj}(A)B^T A^4 (B^2)^{-1}\big) &= \det\big(\operatorname{adj}(A)\big)\det(B^T)\det(A^4)\big(\det(B^2)\big)^{-1} \\ &= \big(\det(A)\big)^3 \det(B)\big(\det(A)\big)^4\big(\det(B)\big)^{-2} \\ &= \big(\det(A)\big)^7 \big(\det(B)\big)^{-1} = (-2)^7 (2)^{-1} = -64. \end{aligned}$$

   (b)

$$\begin{aligned} &\det\big(A^3 (B^2)^T \big((\operatorname{adj}(A))^{-1}\big)^3 B^{-1}\big) \\ &= \big(\det(A)^3 \big(\det(B)\big)^2 \big(\det\big(\operatorname{adj}(A)\big)\big)^{-3}\big(\det(B)\big)^{-1} \\ &= \big(\det(A)^3 \det(B)\big(\det(A)\big)^{-9} = \det(B)\big(\det(A)\big)^{-6} = 2(-2)^{-6} = 1/32. \end{aligned}$$

<div style="text-align: center;">3</div>

6. Note that if A, B and C are $n \times n$ matrices with the same rows except one, say row i, and if row i of C is a sum of row i of A and row i of B, then $\det(C) = \det(A) + \det(B)$. This holds also if rows are replaced by columns.

$$
\begin{aligned}
\det(C) &= \begin{vmatrix} a & b & c \\ 1 & 0 & -3 \\ d & e & f \end{vmatrix} = \begin{vmatrix} a & b & c \\ 3+(-2) & -2+2 & 1+(-4) \\ d & e & f \end{vmatrix} \\[2mm]
&= \begin{vmatrix} a & b & c \\ 3 & -2 & 1 \\ d & e & f \end{vmatrix} + \begin{vmatrix} a & b & c \\ -2 & 2 & -4 \\ d & e & f \end{vmatrix} \\[2mm]
&= \begin{vmatrix} a & b & c \\ 3 & -2 & 1 \\ d & e & f \end{vmatrix} - 2\begin{vmatrix} a & b & c \\ 1 & -1 & 2 \\ d & e & f \end{vmatrix} = \det(B) - 2\det(A) \\[2mm]
&= 5 - 2(4) = 5 - 8 = -3.
\end{aligned}
$$

7. A is invertible iff $0 \neq \det(A) = c^3 - c = c(c-1)(c+1)$. So, A is invertible iff $c \in \mathbb{R} \setminus \{1, 0, -1\}$.

8. (a) $x = \dfrac{\begin{vmatrix} 4 & 2 \\ 13 & 7 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 3 & 7 \end{vmatrix}} = \dfrac{28-26}{7-6} = 2;\ y = \dfrac{\begin{vmatrix} 1 & 4 \\ 3 & 13 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 3 & 7 \end{vmatrix}} = \dfrac{13-12}{7-6} = 1.$

(b) $x = \dfrac{\begin{vmatrix} -3 & -2 & 4 \\ 0 & 3 & 1 \\ 6 & 6 & -5 \end{vmatrix}}{\begin{vmatrix} 3 & -2 & 4 \\ 5 & 3 & 1 \\ 2 & 6 & -5 \end{vmatrix}} = \dfrac{-21}{-21} = 1;\ y = \dfrac{\begin{vmatrix} 3 & -3 & 4 \\ 5 & 0 & 1 \\ 2 & 6 & -5 \end{vmatrix}}{\begin{vmatrix} 3 & -2 & 4 \\ 5 & 3 & 1 \\ 2 & 6 & -5 \end{vmatrix}} = \dfrac{21}{-21} =$

$-1;\ z = \dfrac{\begin{vmatrix} 3 & -2 & -3 \\ 5 & 3 & 0 \\ 2 & 6 & 6 \end{vmatrix}}{\begin{vmatrix} 3 & -2 & 4 \\ 5 & 3 & 1 \\ 2 & 6 & -5 \end{vmatrix}} = \dfrac{42}{-21} = -2.$

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