Practice Problems S6 (Complex Numbers)

1. Write the following complex numbers in the form $a + bi$:

   (a) $\frac{3-i}{2i+5}$, (b) $(2-3i)^3$, (c) $\frac{1-i}{2-3i} - \frac{1+2i}{5+i}$, (d) $e^{5i\pi/3}$.

2. Express the following complex numbers in polar from: (a) $(1-\sqrt{3}i)^5$, (b) $(\sqrt{3}-i)(2-2i)$, (c) $-2e^{\pi i/3}$

3. Prove that $\cos(\theta_1+\theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)$ and $\sin(\theta_1 + \theta_2) = \cos(\theta_1)\sin(\theta_2) - \sin(\theta_1)\cos(\theta_2)$.

4. (a) Express the number $z = (1-i)(-1+\sqrt{3}i)$ in polar form and in the form $a + bi$;

   (b) Find $\cos(5\pi/12)$ and $\sin(5\pi/12)$.

5. Solve the following equations:

   (a) $(i+z) - 3i(2-z) = iz + 1$;

   (b) $z(1+i) = \bar{z} - (3+2i)$;

   (c) $3x^2 + 5x + 10 = 0$;

   (d) $z^2 = -15 - 8i$;

   (e) $z^2 - (3-2i)z + (5-i) = 0$.

6. Solve the following system of linear equations:

$$\begin{cases} x + iy - iz & = 3+i \\ -ix + 2y + iz & = 2 \\ (i-1)x - (1+2i)y + 2z & = i-1 \end{cases}.$$

7. Find the inverse of $A = \begin{bmatrix} 1 & 1-i \\ 2+i & 3+i \end{bmatrix}$.

8. Diagonalize the matrix $A = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$.

9. Find the 8th roots of $z = 128(-1 - \sqrt{3}i)$.

Recommended Problems:

Pages 482 - 483: 1, 2, 3 a, 4 a, b; 5 a, b, c; 6 a, b, d; 10 a, b;11 a, b, c; 18 , 19, 23.

Solutions

1. (a) $\frac{3-i}{2i+5} = \dfrac{(3-i)(5-2i)}{|2i+5|^2} = \dfrac{13-11i}{29} = \dfrac{13}{29} - \dfrac{11}{29}i$; (b) $(2-3i)^3 = -46-9i$; (c)$\frac{3}{26} - \frac{7}{26}i$, (d) $e^{5\pi/3} = \cos(5\pi/3)) + \sin(5\pi/3)i = \dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i.$

2. (a) $(1-\sqrt{3}i)^5 = \big(2(1/2-\sqrt{3}/2i\big)^5 = \big(2e^{-\pi i/3})^5 = 32e^{-5\pi i/3} = 32\big(\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i\big) = 16(1+\sqrt{3}i)$; (b) $(\sqrt{3}-i)(2-2i) = 2e^{-\pi i/6}(2\sqrt{2}e^{-\pi i/4}) = 4\sqrt{2}e^{-i\pi/4-i\pi/6} = 4\sqrt{2}e^{-5\pi i/12}$; (c)$-2e^{\pi i/3} = 2(-1)e^{\pi i/3} = 2e^{\pi i}e^{\pi i/3} = 2e^{(\pi+\pi/3)i} = 2e^{4\pi i/3}$.

3. Use $e^{i\theta_1}e^{i\theta_2} = e^{(\theta_1+\theta_2)i}$ (Multiplication rule). It follows that

$$\Big(\cos(\theta_1) + i\sin(\theta_1)\Big)\Big(\cos(\theta_2) + i\sin(\theta_2)\Big) = \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)$$
$$\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) + i\Big(\cos(\theta_1)\sin(\theta_2) + \sin(\theta_1\cos(\theta_2)\Big)$$
$$= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2).$$

Therefore, $\cos(\theta_1+\theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)$ and $\sin(\theta_1+\theta_2) = \cos(\theta_1)\sin(\theta_2) + \sin(\theta_1\cos(\theta_2)$.

4. (a) $z = (1-i)(-1+\sqrt{3}i) = 2\sqrt{2}e^{-\pi i/4}e^{2\pi i/3} = 2\sqrt{2}e^{-\pi i/4+2\pi i/3} = 2\sqrt{2}e^{5\pi i/12}$; (b) Since $2\sqrt{2}e^{5\pi i/12} = z = (1-i)(-1+\sqrt{3}i) = (-1+\sqrt{3})+(1+\sqrt{3})i$, we have $\cos(5\pi i/12) = \dfrac{\sqrt{2}(\sqrt{3}-1)}{4}$ and $\sin(5\pi i/12) = \dfrac{\sqrt{2}(1+\sqrt{3})}{4}$.

5. (a) $z = 11/5 + 3i/5$; (b) $z = -8+3i$; (c) $x = -5/6 - i\sqrt{95}/6$ or $x = -5/6 + i\sqrt{95}/6$; (d) $z = 1-4i$ or $z = -1+4i$; (e) $z = 2-3i$ or $z = 1+i$.

6. Carry the augmented matrix to reduced row-echelon form:

$$\begin{bmatrix} 1 & 0 & 1-2\,i & 6 \\ 0 & 1 & 1+i & 1+3\,i \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which gives $x = 6 - (1-2i)t$, $y = 1 + 3i - (1+i)t$ and $z = t$, where $t \in \mathbb{C}$.

7. A has inverse $\begin{bmatrix} 1/2 - 3/2\, i & 1/2 + 1/2\, i \\ -1/2 + i & -1/2\, i \end{bmatrix}$.

8. A has eigenvalues $\lambda_1 = 1 - i$ and $\lambda_1 = 1 + i$ and corresponding basic eigenvectors $X_1 = [-1\ \ 1]^T$ and $X_2 = [1\ \ 1]^T$. Therefore, the matrix $P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ diagonalizes A, i.e., $P^{-1}AP = \mathrm{diag}(1-i, 1+i)$.

9. The complex number $z = 128(-1-\sqrt{3}i) = 256e^{4\pi i/3} = 2^8 e^{4\pi i/3}$ has 8th roots given by $z_k = 256^{1/8} e^{\overline{\frac{4\pi/3 + 2k\pi)i}{8}}} = 2e^{(\pi/6 + k\pi/4)i}$, $k = 0, 1, 2, 3,.$ So, $z_0 = \mathbf{2e^{\pi/6i}} = 2(\cos(\pi/6) + i\sin(\pi/6)) = \boldsymbol{\sqrt{3} + i}$;
$z_1 = 2e^{(\pi/6+\pi/4)i} = \mathbf{2e^{5\pi i/12}} = \dfrac{\sqrt{2}}{2}(\sqrt{3} - 1 + i\left(1 + \sqrt{3}\right))$;
$z_2 = 2e^{(\pi/6+2\pi/4)i} = \mathbf{2e^{2\pi i/3}} = -1 + \sqrt{3}i$;
$z_3 = 2e^{(\pi/6+3\pi/4)i} = \mathbf{2e^{11\pi i/12}} = \dfrac{\sqrt{2}}{2}(-\sqrt{3} - 1 + i\left(-1 + \sqrt{3}\right))$;
$z_4 = 2e^{(\pi/6+4\pi/4)i} = \mathbf{2e^{7\pi i/6}} = -\sqrt{3} - i$;
$z_5 = 2e^{(\pi/6+5\pi/4)i} = \mathbf{2e^{17\pi i/12}} = \dfrac{\sqrt{2}}{2}(-\sqrt{3} + 1 - i\left(1 + \sqrt{3}\right))$;
$z_6 = 2e^{(\pi/6+6\pi/4)i} = \mathbf{2e^{5\pi i/3}} = -1 + \sqrt{3}i$;
$z_7 = 2e^{(\pi/6+7\pi/4)i} = \mathbf{2e^{23\pi i/12}} = \dfrac{\sqrt{2}}{2}(\sqrt{3} + 1 + i\left(1 - \sqrt{3}\right))$.