Practice Problems S7 (Vector Geometry)

1. Let $P_1(2,1,-2)$ and $P_2(1,-2,0)$ be points in $\mathbb{R}^3$.
   (a) Find the parametric equations of the line through P_1 and P_2;
   (b) Find the coordinates of the point P that is $1/4$ of the way from P_1 to P_2.

2. Find the point of intersection P between the lines (if they are concurrent):
$$\begin{cases} x = 3 + t \\ y = 2 + 3t \\ z = -1 - 3t \end{cases} \quad \text{and} \quad \begin{cases} x = 1 - s \\ y = 1 + 2s \\ z = 3 + s \end{cases}.$$

3. Find the equation of the plane passing through the point $P(3,-7,5)$ and is perpendicular to the line $\begin{cases} x = 2 + 6t \\ y = -5 - 6t \\ z = 3 + 5t \end{cases}$

4. Find the equation of the plane through the points $A(3,-7,1)$, $B(2,0,-1)$ and $C(1,3,0)$. Check if the point $D(5,1,1)$ lies on this plane.

5. Determine whether the plane $2x-3y+z=1$ contains the line $\begin{cases} x = 3 + 2t \\ y = 2 \\ z = 1 - 4t \end{cases}$.

6. Find the line of intersection of the planes $(\pi_1) \equiv 3x + 5y + 4z = 5$ and $(\pi_2) \equiv x + 2y + 3z = 2$.

7. Find the shortest distance from the point $P(1,1,1)$ to the line $\begin{cases} x = 3 + t \\ y = 9 \\ z = 10 - 4t \end{cases}$.

   Which point on this line is closest to P?

8. Find the shortest distance from the point $P(4, 1, 9)$ to the plane $x - 4z = 2$. Which point on this plane is closest to P?

9. Let $\overrightarrow{u} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $\overrightarrow{v} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, $\overrightarrow{w} = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$ be vectors in $\mathbb{R}^3$. Compute:

   (a) $\overrightarrow{v} \times \overrightarrow{w}$ and then $\overrightarrow{u} \times (\overrightarrow{v} \times \overrightarrow{w})$;

   (b) $(\overrightarrow{u} \cdot \vec{w})\, \overrightarrow{v} - (\overrightarrow{u} \cdot \overrightarrow{v})\, \overrightarrow{w}$.

10. Find the areas of the sides of the parallelepiped determined by the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{AD}$, where A, B, C and D are the points in Problem 4. What is the volume of this parallelepiped?

Recommended Problems:

Pages 165 - 167: 1 a, c; 3 a; 4 a; 5, 7 a; 9 a, b; 15, 20, 24
Pages 177 - 179: 1 a; 2 a, b; 3 a; 6, 8, 9, 10 a; 11 a; 12, 13, 14, 15, 16 a; 18, 19, 24 a
Page 185: 3 a; 4 a; 5 a.

Solutions

1. (a) If $P(x, y, z)$ is a point on the line though $P1$ and $P2$, then there is $t \in \mathbb{R}$ such taht $\overrightarrow{P1P} = t\,\overrightarrow{P_1P_2} = t[-1 \;\; -3 \;\; 2]^T$. Thus, $\begin{cases} x = 2 - t \\ y = 1 - 3t \\ z = -2 + 2t \end{cases}$.

   (b) $\overrightarrow{P1P} = \dfrac{1}{4}\,\overrightarrow{P_1P_2}$. This gives $P(7/4, 1/4, -3/2)$.

2. Solve the following system of linear equations $\begin{cases} x = 3 + t = 1 - s \\ y = 2 + 3t = 1 + 2s \\ z = -1 - 3t = 3 + s \end{cases}$ to get $t = -1 = s$. The point of intersection P has coordinates $P(2, -1, 2)$.

3. The plane has normal vector $\vec{v} = [6 \;\; -6 \;\; 5]^T$ (direction vector of the given line). So, the scalar equation is $6(x-3) - 6(y+7) + 5(z-5) = 0$.

4. If $P(x, y, z)$ is a point on the plane through A, B and C, then the equation of the plane is

$$\begin{aligned} 0 &= \overrightarrow{AP} \cdot (\overrightarrow{AB} \times \overrightarrow{AC}) = \det(\overrightarrow{AP}, \overrightarrow{AB}, \overrightarrow{AC}) = \begin{vmatrix} x-3 & y+7 & z-1 \\ 2-3 & 0+7 & -1-1 \\ 1-3 & 3+7 & 0-1 \end{vmatrix} \\ &= 13(x-3) + 3(y+7) + 4(z-1). \end{aligned}$$

   Replacing x, y and z by the coordinates of $D(5, 1, 1)$, we have: $13(5 - 3) + 3(1+7) + 4(1-1) = 26 + 24 = 50 \neq 0$. Therefore this plane does not contain the point D.

5. A line can be completely on the given plane, parallel to the plane or can intersect the plane at one point. Replace the parametric expressions of x, y and z into the equation of the plane. If one gets a consistent equation in the parameter t, then the line intersects the plane. If the parameter t is cancels out, then the line is contained in the plane if the equation is consistent, otherwise, the line is parallel to the plane. With $x = 3+2t$, $y = 2$ and $z = 1-4t$, we have: $2(3+2t) - 3(2) + (1-4t) = 1$, i.e. $1 = 1$, this implies that the plane contains the line.

6. Solve $\left\{ \begin{array}{c} 3x+5y+4z=5 \\ x+2y+3z=2 \end{array} \right.$ to find the line of intersection $\left\{ \begin{array}{c} x=7t \\ y=1-5t \\ z=t \end{array} \right.$.

7. The line has direction vector $\vec{d}=[1\ \ 0\ \ -4]^T$. Choose arbitrarily a point $P_0=(3,9,10)$ on the line. Project the vector $\overrightarrow{P_0P}=[-2\ \ -8\ \ -9]^T$ on $\overrightarrow{d}$: $\mathrm{proj}_{\overrightarrow{d}}\overrightarrow{P_0P}=\dfrac{\overrightarrow{P_0P}\cdot\overrightarrow{d}}{\|\overrightarrow{d}\|^2}\overrightarrow{d}=\dfrac{-2+36}{17}\vec{d}=2\vec{d}$. The shortest distance is $\|\overrightarrow{P_0P}-\mathrm{proj}_{\overrightarrow{d}}\overrightarrow{P_0P}\|=\|\overrightarrow{P_0P}-2\overrightarrow{d}\|=\sqrt{4^2+8^2+1^2}=9$. The closest point $Q(x,y,z)$ is given by $\overrightarrow{P_0Q}=\mathrm{proj}_{\overrightarrow{d}}\overrightarrow{P_0P}$. So, $Q(5,9,2)$.

8. The plane has normal vector $\vec{n}=[1\ \ 0\ \ -4]^T$. Choose arbitrarily a point $P_0(6,0,1)$ on the plane. Project the vector $\overrightarrow{P_0P}=[-2\ \ 1\ \ 8]^T$ on $\vec{n}$: The shortest distance is given by $\|\mathrm{proj}_{\vec{n}}\overrightarrow{P_0P}\|=\|-2\overrightarrow{n}\|=2\sqrt{17}$. The closest point point $Q(x,y,z)$ is given by $\overrightarrow{QP}=\mathrm{proj}_{\vec{n}}\overrightarrow{QP}=-2\vec{n}$. So, $Q(6,1,1)$.

9. **(a)**

$$\overrightarrow{v}\times\overrightarrow{w}=\begin{vmatrix}\overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k}\\ 1 & -2 & 3\\ -3 & 1 & 2\end{vmatrix}=\begin{vmatrix}-2 & 3\\ 1 & 2\end{vmatrix}\overrightarrow{i}-\begin{vmatrix}1 & 3\\ -3 & 2\end{vmatrix}\overrightarrow{j}$$

$$+\begin{vmatrix}1 & -2\\ -3 & 1\end{vmatrix}\overrightarrow{k}=-7\overrightarrow{i}-11\overrightarrow{j}-5\overrightarrow{k}=\begin{bmatrix}-7\\ -11\\ -5\end{bmatrix}.$$

And $\overrightarrow{u}\times(\overrightarrow{v}\times\overrightarrow{w})=\begin{vmatrix}\overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k}\\ 1 & 2 & 2\\ -7 & -11 & -5\end{vmatrix}=12\overrightarrow{i}-9\overrightarrow{j}+3\overrightarrow{k}=$ $\begin{bmatrix}12\\ -9\\ 3\end{bmatrix}$.

(b) $(\overrightarrow{u}\cdot\overrightarrow{w})\,\overrightarrow{v}-(\overrightarrow{u}\cdot\overrightarrow{v})\,\overrightarrow{w}=(-3+2+4)\overrightarrow{v}-(1-4+6)\overrightarrow{w}=\begin{bmatrix}12\\ -9\\ 3\end{bmatrix}$.

Note: In general, $\overrightarrow{u} \times (\overrightarrow{v} \times \overrightarrow{w}) = (\overrightarrow{u} \cdot \vec{w})\, \overrightarrow{v} - (\overrightarrow{u} \cdot \overrightarrow{v})\, \overrightarrow{w}$. But $\overrightarrow{u} \times (\overrightarrow{v} \times \overrightarrow{w}) \neq (\overrightarrow{u} \times \overrightarrow{v}) \times \overrightarrow{w}$, i.e., the cross product is not associative.

10. The parallelepiped has six sides (2 times the parallelograms determined by the vectors). So, these sides have areas: $\|\overrightarrow{AB} \times \overrightarrow{AC}\| = \|[13\ \ 3\ \ 4]^T\| = \sqrt{194}$, $\|\overrightarrow{AB} \times \overrightarrow{AD}\| = \|[16\ \ -4\ \ -22]^T\| = \sqrt{756}$ and $\|\overrightarrow{AC} \times \overrightarrow{AD}\| = \|[8\ \ -2\ \ -36]^T\| = 2\sqrt{341}$. The volume of this parallelepiped is $Vol = |\overrightarrow{AD} \cdot (\overrightarrow{AB} \times \overrightarrow{AC})| = |\det(\overrightarrow{AD}, \overrightarrow{AB}, \overrightarrow{AC})| = 50$.