Problem 1. (a) $K = [A|B] = \left[\begin{array}{cccc|c} 2 & 7 & -8 & 3 & -4 \\ -1 & -3 & 3 & -1 & 1 \\ 2 & 3 & 0 & -1 & 4 \end{array}\right]$

(b) $[K|I_3] \longrightarrow [R|U]$:

$R = \left[\begin{array}{cccc|c} 1 & 0 & 3 & -2 & 5 \\ 0 & 1 & -2 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$; $U = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 3 & 8 & 1 \end{bmatrix}$, ($U$ is not unique, e.g., $U = \begin{bmatrix} -3 & -7 & 0 \\ 1 & 2 & 0 \\ 3 & 8 & 1 \end{bmatrix}$, or a different one).

(c) Gen. sol. to the syst.: $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$; Particular solution: $\begin{bmatrix} 5 \\ -2 \\ 0 \\ 0 \end{bmatrix}$; Gen. sol. to the associated hom. syst.: $X_0 = t \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$; Basic solutions: $\begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

Problem 2. (a) $\begin{bmatrix} 2 & 5 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$; $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 3 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ -5 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. ($x = -3$; $y = 2$).

(b) $A = -\frac{1}{11} \begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix}$.

Problem 3. $[A|I_3] \longrightarrow [I_3|A^{-1}]$: $A^{-1} = \begin{bmatrix} 4 & -5 & -1 \\ -3 & 5 & -1 \\ 1 & -2 & 1 \end{bmatrix}$.

Problem 4. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. (This product is not unique.)

Problem 5. (a) $a = -x + 2y$ and $b = x - y$;

(b) $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = (-x+2y)\begin{bmatrix} -1 \\ 2 \end{bmatrix} + (x-y)\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2x - 3y \\ -3x + 5y \end{bmatrix}$;

(c) $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$;

(d) $A_T = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$

Problem 6. (a) $S_0 = [1\ 0]^T$, $S_2 = PS_0$, $S_2 = PS_1 = PPS_0 = P^2S_0 = [9/25\ \mathbf{16/25}]^T$. Therefore, the probability to be in state 2 after 2 transitions is 16/25.

(b) Since $P^1 = P$ has no zero entries, the chain is regular. Its steady state-vector S is the only probability vector which solves the homogeneous system $(I_2 - P)X = 0$. $S = [1/3\ 2/3]^T$.