UNIVERSITY OF CALGARY

DEPARTMENT OF MATHEMATICS AND STATISTICS

MATHEMATICS 211 — L20 Spring 2009

MIDTERM EXAM [June 15, 2009 (Monday)]

Time: 75 minutes. PLEASE write your Name on the very last page.
NO CALCULATORS.
Total Marks = 100. Work all problems. Marks are shown in brackets.

Student ID: ____________________

[Marks]

1. Let

$$\begin{cases} 2x_1 + 7x_2 - 8x_3 + 3x_4 & = & -4 \\ -x_1 - 3x_2 + 3x_3 - x_4 & = & 1 \\ 2x_1 + 3x_2 + 0x_3 - x_4 & = & 4 \end{cases}.$$

[2] (a) Write down the augmented matrix $K = [A|B]$ of the system;

[14] (b) Find the reduced row-echelon form R of $K = [A|B]$ and an invertible matrix U of size 3×3 such that the product $U\,K = R$;

[14] (c) Use the reduced row-echelon form R of $K = [A|B]$ from part (b) to find the general solution to this system. Give a particular solution and the general solution to the associated homogeneous system with basic solutions. (Your answers should be in matrix form)

[8] 2. (a) Write the following system in matrix form and use a matrix inverse to solve it:

$$\left\{ \begin{array}{rcr} 2x + 5y & = & 4 \\ 3x + 2y & = & -5 \end{array} \right. .$$

[4] (b) Find A if $(A^{-1} - 2I_2)^T = \left[\begin{array}{cc} 0 & 3 \\ 5 & 0 \end{array} \right]$.

[18] 3. Use row operations to find the inverse A^{-1} of $A = \begin{bmatrix} 3 & 7 & 10 \\ 2 & 5 & 7 \\ 1 & 3 & 5 \end{bmatrix}$.

[12] 4. Express the matrix $A = \begin{bmatrix} 4 & 9 \\ 1 & 2 \end{bmatrix}$ as a product of elementary matrices.

5. Let $T\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a linear transformation of vectors in $\mathbb{R}^2$ such that $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Write your answers in the space provided.

(a) Express the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$:

$$\begin{bmatrix} x \\ y \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

[4] where $a =$ __________ and $b =$ __________. (a and b depend on x and y)

[4] (b) Find a formula for $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$:

$$\begin{aligned} T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= a\, T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) + b\, T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) \\ &= \_\_\_\_\_\_ \begin{bmatrix} \_\_\_ \\ \_\_\_ \end{bmatrix} + \_\_\_\_\_\_ \begin{bmatrix} \_\_\_ \\ \_\_\_ \end{bmatrix} \\ &= \begin{bmatrix} \_\_\_\_\_\_ \\ \_\_\_\_\_\_ \end{bmatrix}; \end{aligned}$$

[2] (c) Use the formula from part (b) to find

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \_\_\_ \\ \_\_\_ \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} \_\_\_ \\ \_\_\_ \end{bmatrix};$$

[2] (d) The matrix of T is

$$A_T = \begin{bmatrix} \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ \end{bmatrix}.$$

6. Let $P = \begin{bmatrix} 1/5 & 2/5 \\ 4/5 & 3/5 \end{bmatrix}$ be the transition matrix of a Markov chain that starts in state 1.

[6] (a) What is the probability that the chain is in state 2 after 2 transitions?

[10] (b) Explain why this chain is regular and find the steady-state vector for the chain.

Name:	Student ID:	Marks: