1. (a) The system $\left\{\begin{array}{lcl} x + a\,y &=& 1 \\ a\,x + 4\,y &=& 2 \end{array}\right.$ has coefficient matrix $A = \begin{bmatrix} 1 & a \\ a & 4 \end{bmatrix}$ and augmented matrix $A = \left[\begin{array}{cc|c} 1 & a & 1 \\ a & 4 & 2 \end{array}\right]$.

(b) $A = \left[\begin{array}{cc|c} 1 & a & 1 \\ a & 4 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & a & 1 \\ 0 & 4-a^2 & 2-a \end{array}\right] (r_2 - a \times r_1 \text{ into } r_2)$.

Case 1: If $4 - a^2 \neq 0$, i.e., $a \neq \pm 2$, then divide row 2 by $4 - a^2$ to get:

$$\longrightarrow \left[\begin{array}{cc|c} 1 & a & 1 \\ 0 & 1 & \frac{2-a}{4-a^2} \end{array}\right] = \left[\begin{array}{cc|c} 1 & a & 1 \\ 0 & 1 & \frac{1}{2+a} \end{array}\right].$$

Back substitution: $y = \frac{1}{2+a}$ and $x = 1 - a\,y = 1 - \frac{a}{2+a} = \frac{2}{2+a}$. In this case, the system has a unique solution given by $x = \frac{2}{2+a}$ and $y = \frac{1}{2+a}$.

Case 2: If $4 - a^2 = 0$, then $a = 2$ or $a = -2$. For $a = 2$, we have: $\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & 0 \end{array}\right]$, and the system has infinitely many solutions given by $\left\{\begin{array}{lcl} x &=& 1-2t \\ y &=& t \end{array}\right.$, $t \in \mathbb{R}$. Finally, for $a = -2$ we have $\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 0 & 4 \end{array}\right]$, and the system is inconsitent, i.e., has no solutions.

2. (a) Replacing r_2 by $r_2 - 3 \times r_1$ and r_3 by $r_3 + r_1$:

$$A = \begin{bmatrix} 1 & -1 & 3 & 5 \\ 3 & -2 & 1 & -2 \\ -1 & 1 & 1 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 3 & 5 \\ 0 & 1 & -8 & -17 \\ 0 & 0 & 4 & 8 \end{bmatrix}$$

then replacing r_3 by $\frac{1}{4} r_3)$:

$$\longrightarrow \begin{bmatrix} \mathbf{1} & -1 & 3 & 5 \\ 0 & \mathbf{1} & -8 & -17 \\ 0 & 0 & \mathbf{1} & 2 \end{bmatrix}.$$

(b) The last matrix in part (a) is in row echelon form (this is not unique: a different choice of row operation may lead to a different row-echelon form) with 3 leading 1's. Therefore, A has rank 3.

(c) The system is equivalent to $\left\{\begin{array}{lcl} x_1 - x_2 + 3x_3 &=& 5 \\ x_2 - 8x_3 &=& -17 \\ x_3 &=& 2 \end{array}\right.$. Back substitution: $x_3 = 2$, $x_2 = -1 + 8x_3 = -17 + 16 = -1$ and

$x_1 = 5 + x_2 - 3x_3 = 5 - 1 - 6 = -2$. So, the system has a unique solution $x_1 = -2$, $x_2 = -1$ and $x_3 = 2$.

3. (a) Row operations bring the matrix to its reduced row-echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 2 & -2 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

   (b) Use the reduced row-echelon form from part (a) as the augmented matrix (the last column is the constant matrix) of an equivalent system. With $x_4 = t$, $t \in \mathbb{R}$, the general solution to the system is : $x_1 = 4$, $x_2 = -2 - 2t$, $x_3 = -1 - t$, $x_4 = t$.

4. The homogeneous system has coefficient matrix $A = \begin{bmatrix} 1 & -1 & 3 & 1 \\ -1 & -2 & 6 & 2 \\ 2 & 1 & 3 & 5 \\ 2 & -2 & 12 & 8 \end{bmatrix}$.

   Its reduced row-echelon form is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Set $x_4 = t$. Then $x_3 = -t$, $x_2 = -2t$ and $x_1 = 0$, $t \in \mathbb{R}$.