Practice Problems S2

Solutions

1. X is a linear combination of vectors X_1 and X_2 if $X = aX_1 + bX_2$ for some numbers a and b:
$$\begin{bmatrix} -1 \\ 2 \end{bmatrix} = a\begin{bmatrix} 2 \\ 3 \end{bmatrix} + b\begin{bmatrix} 3 \\ 4 \end{bmatrix} \Longleftrightarrow \begin{cases} 2a + 3b &= -1 \\ 3a + 4b &= 2 \end{cases}$$
Solve this linear system in a and b to get $a = 10$ and $b = -7$, ie., X is a linear combination of X_1 and X_2 with $X = 10X_1 - 7X_2$.

2. X_1, X_2 and X_3 are linearly independent if the equation $aX_1 + bX_2 + cX_3 = 0$ is satisfied only if $a = b = c = 0$. Let $a, b, c \in \mathbb{R}$ such that $aX_1 + bX_2 + cX_3 = 0$, i.e., $a\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + b\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = 0$. Then a, b, c are solutions to the homogeneous system $\begin{cases} 2a + b - c &= 0 \\ -a + \quad +c &= 0 \\ 3a + 2b + c &= 0 \end{cases}$. The coefficient matrix (homogeneous system in **3** variables) has rank **3**, therefore the system has a unique solution (the trivial one) $a = b = c = 0$. So these vectors are lin. ind.

3. The system has augmented matrix $\left[\begin{array}{cccc|c} 1 & -2 & 1 & -4 & 1 \\ 1 & 3 & 7 & 2 & 2 \\ 1 & -12 & -11 & -16 & -1 \end{array}\right]$ which can be brought by row operations to the reduced row-echelon matrix $\left[\begin{array}{cccc|c} 1 & 0 & 17/5 & -8/5 & 7/5 \\ 0 & 1 & 6/5 & 6/5 & 1/5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$.

The system has general solution
$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7/5 - 17/5t + 8/5s \\ 1/5 - 6/5t - 6/5s \\ t \\ s \end{bmatrix} = \begin{bmatrix} 7/5 \\ 1/5 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -17/5 \\ -6/5 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 8/5 \\ -6/5 \\ 0 \\ 1 \end{bmatrix}.$$

$X_1 = \begin{bmatrix} 7/5 \\ 1/5 \\ 0 \\ 0 \end{bmatrix}$ is a particular solution to the system. The associated homogeneous system has general solution $X_0 = t\begin{bmatrix} -17/5 \\ -6/5 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 8/5 \\ -6/5 \\ 0 \\ 1 \end{bmatrix}$, with $X_2 = \begin{bmatrix} -17/5 \\ -6/5 \\ 1 \\ 0 \end{bmatrix}$ and $X_3 = \begin{bmatrix} 8/5 \\ -6/5 \\ 0 \\ 1 \end{bmatrix}$, or $X_4 = \begin{bmatrix} -17 \\ -6 \\ 5 \\ 0 \end{bmatrix}$ and $X_5 = \begin{bmatrix} 8 \\ -6 \\ 0 \\ 5 \end{bmatrix}$ as basic solutions.

4. The augmented matrix of the system has reduced row-echelon form

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & -4 & -9 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Therefore, the general solution is

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3-t \\ -9+4t \\ -2+t \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ -9 \\ -2 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 4 \\ 1 \\ 1 \end{bmatrix}.$$

$X_1 = \begin{bmatrix} 3 \\ -9 \\ -2 \\ 0 \end{bmatrix}$ is a particular solution. The associated homogeneous system has general solution $X_0 = t\begin{bmatrix} -1 \\ 4 \\ 1 \\ 1 \end{bmatrix}$ with $\begin{bmatrix} -1 \\ 4 \\ 1 \\ 1 \end{bmatrix}$ as a basic solution.

5. $AB = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & 4 \end{bmatrix}\begin{bmatrix} 2 & 3 & 1 \\ 1 & 9 & 7 \\ -1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -6 & -2 \\ 0 & 6 & 10 \end{bmatrix}$,

$$BA^T = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 9 & 7 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 8 \\ 6 & 30 \\ 3 & 6 \end{bmatrix},$$

$$BC = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 9 & 7 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 1 & 1 & 1 \\ -1 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 13 & 8 \\ 3 & 40 & 18 \\ -3 & 5 & 0 \end{bmatrix},$$

$$CB = \begin{bmatrix} 1 & 3 & 2 \\ 1 & 1 & 1 \\ -1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 1 & 9 & 7 \\ -1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 30 & 26 \\ 2 & 12 & 10 \\ 1 & 33 & 29 \end{bmatrix}.$$

6. This directed graph has adjacency matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. The number of 5-paths (i.e., paths of length 5) from a vertex v_j to another vertex v_i is the (i, j)-entry of $A^5 = \begin{bmatrix} 5 & 3 & \mathbf{5} \\ 5 & 3 & 5 \\ 3 & \mathbf{2} & 3 \end{bmatrix}$. Therefore, there are 2 5-paths from v_2 to v_3 and 5 5-paths from v_3 to v_1.