Practice Problems S3 (Solutions)

1. A and B are square matrices with $AB = I_2$, they are inverses of each other.

2. (a) $\det(A) = \begin{vmatrix} 7 & 4 \\ 3 & 2 \end{vmatrix} = 7 \times 2 - 3 \times 4 = 2 \neq 0$. So, A is invertible. Its inverse $A^{-1} = \dfrac{1}{\det(A)}\text{adj}(A) = \dfrac{1}{2}\begin{bmatrix} 2 & -4 \\ -3 & 7 \end{bmatrix}$. It can be also obtained by bringing $[A|I_2]$ to the reduced row-echelon form $[I_2|A^{-1}]$.

   (b) By row operations carry the matrix $[A|I_3] = \left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ -1 & 4 & 1 & 0 & 0 & 1 \end{array}\right]$ to its reduced row-echelon form ($[A|I_3] \longrightarrow [I_3|A^{-1}]$):

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 5 & 1 \\ 0 & 1 & 0 & -2 & 3 & 1 \\ 0 & 0 & 1 & 5 & -7 & -2 \end{array}\right].$$

   Therefore, $A^{-1} = \begin{bmatrix} -3 & 5 & 1 \\ -2 & 3 & 1 \\ 5 & -7 & -2 \end{bmatrix}$.

3. In matrix form:

   (a)

$$\begin{bmatrix} 7 & 4 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} \implies \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ 3 & 2 \end{bmatrix}^{-1}\begin{bmatrix} 2 \\ -2 \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} 2 & -4 \\ -3 & 7 \end{bmatrix}\begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 6 \\ -10 \end{bmatrix},$$

   i.e., $x = 6$ and $y = -10$.

(b)

$$\begin{bmatrix} 1 & 3 & 2 \\ 1 & 1 & 1 \\ -1 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 5 \end{bmatrix} \implies \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 3 & 2 \\ 1 & 1 & 1 \\ -1 & 4 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 1 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 5 & 1 \\ -2 & 3 & 1 \\ 5 & -7 & -2 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} -5 \\ -2 \\ 8 \end{bmatrix},$$

i.e., $x = -5$, $y = -2$ and $z = 8$.

4. (a) Take the inverses of both sides: $A \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}^{-1} \Rightarrow$ $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}^{-1} = \left(\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \right)^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} =$ $\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$;

   (b) Transpose both sides: $A^{-1} - 2I_2 = -2 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow A^{-1} = 2I_2 -$ $\begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ -2 & 2 \end{bmatrix} \Rightarrow A = \begin{bmatrix} -1/2 & -1/2 \\ -1/2 & 0 \end{bmatrix}$.

5. Exercise.

6. To find such an invertible matrix U that transforms A into its reduced row-echelon form R, we have to bring $[A|I_3]$ to $[R|U]$:

$$[A|I_3] = \left[\begin{array}{cccc|ccc} \mathbf{1} & -1 & 3 & 5 & 1 & 0 & 0 \\ 3 & -2 & 1 & -2 & 0 & 1 & 0 \\ -1 & 1 & 1 & 3 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|ccc} \mathbf{1} & -1 & 3 & 5 & 1 & 0 & 0 \\ 0 & \mathbf{1} & -8 & -17 & -3 & 1 & 0 \\ 0 & 0 & 4 & 8 & 1 & 0 & 1 \end{array}\right]$$

(Row 2 has been replaced by row2-3xrow1, and Row 3 by $r3 + r1$)

$$r3 \text{ by } \frac{r3}{4} \longrightarrow \left[\begin{array}{cccc|ccc} \mathbf{1} & -1 & 3 & 5 & 1 & 0 & 0 \\ 0 & \mathbf{1} & -8 & -17 & -3 & 1 & 0 \\ 0 & 0 & \mathbf{1} & 2 & 1/4 & 0 & 1/4 \end{array}\right]$$

$$r1 \text{ by } r1{-}3(r3) \text{ and } r2 \text{ by } r2{+}8(r3) \longrightarrow \left[\begin{array}{cccc|ccc} \mathbf{1} & -1 & 0 & -1 & 1/4 & 0 & -3/4 \\ 0 & \mathbf{1} & 0 & -1 & -1 & 1 & 2 \\ 0 & 0 & \mathbf{1} & 2 & 1/4 & 0 & 1/4 \end{array}\right]$$

$$r1 \text{ by } r1 + r2 \longrightarrow \left[\begin{array}{cccc|ccc} \mathbf{1} & 0 & 0 & -2 & -3/4 & 1 & 5/4 \\ 0 & \mathbf{1} & 0 & -1 & -1 & 1 & 2 \\ 0 & 0 & \mathbf{1} & 2 & 1/4 & 0 & 1/4 \end{array}\right].$$

So, $R = \begin{bmatrix} \mathbf{1} & 0 & 0 & -2 \\ 0 & \mathbf{1} & 0 & -1 \\ 0 & 0 & \mathbf{1} & 2 \end{bmatrix}$ and $U = \begin{bmatrix} -3/4 & 1 & 5/4 \\ -1 & 1 & 2 \\ 1/4 & 0 & 1/4 \end{bmatrix}$. One can check that $R = UA$. Note that this invertible matrix U is not unique. It depends on the sequence of row operations performed to bring A to its reduced row-echelon form.

7. The matrix $A = \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}$ has determinant $\det(A) = 5 \times 1 - 2 \times 3 = -1 \neq 0$. So, A is invertinble. It can be carried to the identity matrix I_2 by row operations.

   Step 1: Substract 2 times row 2 from row 1:

   $$A = \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \quad \left(I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow F_1 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\right)$$

   Step 2: Substract 2 times row 1 from row 2:

   $$\longrightarrow \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \quad \left(I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow F_2 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\right)$$

   Step 3: Multiply row 2 by -1:

   $$\longrightarrow \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \left(I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow F_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right)$$

   Step 4: Substract row 2 from row 1:

   $$\longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \left(I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow F_4 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\right)$$

   We have $A^{-1} = F_4F_3F_2F_1$. Therefore, $A = E_1E_2E_3E_4$, where E_1, E_2, E_3 and E_4 are elementary matrices given by $E_1 = F_1^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $E_2 = F_2^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, $E_3 = F_3^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $E_4 = F_4^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

8. Denote by $\vec{v}'$ the reflection of $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ in the line $y = x$ and $\vec{v}''$ the reflection of $\vec{v}'$ in the line $y = -x$. We know that $\vec{v}' = \begin{bmatrix} y \\ x \end{bmatrix}$. From the figure, we see that $\vec{v}''$ is the reflection of $\vec{v}'$ about the origin. Therefore, $\vec{v}'' = \begin{bmatrix} -y \\ -x \end{bmatrix}$. It follows that the reflection about the line $y = -x$ is a linear transformation with matrix $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$. It is the composition of the reflection in the line $y = x$ with the reflection about the origin: $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$.

Figure 1: Reflection of vectors of $\mathbb{R}^2$ about the line $y = -x$:

9. (a) The reflection of vectors of $\mathbb{R}^2$ in the y-axis is defined by $T([x\ \ y]^T) = [-x\ \ y]^T$ and the rotation $R_{\pi/2}$ is given by $R_{\pi/2}([x\ \ y]^T) = [-y\ \ x]$. We have: $R_{\pi/2} \circ T([x\ \ y]^T) = R_{\pi/2}(T([x\ \ y]^T)) = R_{\pi/2}([-x\ \ y]^T) = [-y\ \ -x]^T$. Therefore, the reflectionin the y-axis followed by the rotation through $\pi/2$ is the reflection about the line $y = x$.

   (b) $R_{\pi/2}([x\ \ y]^T)$ and $T([x\ \ y]^T) = [y\ \ x]^T$ is the reflection about the line $y = x$. We have $T \circ R_{\pi/2}([x\ \ y]^T) = T(R_{\pi/2}([x\ \ y]^T)) = T([-y\ \ x]^T) = [x\ \ -y]^T$. It is the reflection in the x-axis.

10. Given that $T([1\ \ -2]^T) = [3\ \ 4]^T$ and $T([-2\ \ 5]^T) = [-1\ \ 4]^T$, $T([-4\ \ 3]^T)$ can be computed if we can express $[-4\ \ 3]^T$ as a linear combination of vectors $[1\ \ -2]^T$ and $[-2\ \ 5]^T$, i.e.; find a and b such that $[-4\ \ 3]^T =$

$a[1 \ \ -2]^T + b[-2 \ \ 5]^T$. We have:

$$\begin{cases} -4 &= a - 2b \\ 3 &= -2a + 5b \end{cases} \implies \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

$$\implies \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}^{-1} \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

$$= \frac{1}{5-4} \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -4 \\ 3 \end{bmatrix} = \begin{bmatrix} -14 \\ -5 \end{bmatrix},$$

i.e., $a = -14$ and -5. It follows that

$$T([-4 \ \ 3]^T) = aT([1 \ \ -2]^T) + bT([-2 \ \ 5]^T) = -14[3 \ \ 4]^T - 5[-1 \ \ 4]^T = [-37 \ \ 76]^T.$$