Practice Problems S4 (Solutions)

1. This Markov chain with transition matrix $P = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$, starting in state 1, has initial state vector $S_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

   (a) Since $P^1 = P$ is the transition matrix with only nonzero entries, all powers of P also have nonzero entries. So, at least one power of P has nonzero entries, i.e., the chain is regular.

   (b) $S_{m+1} = PS_m = P^{m+1}S_0$, where $S_0 = [1\ 0]$. So, $S_1 = PS_0 = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$ and the state-vector S_2 after two transitions is $S_2 = PS_1 = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \mathbf{\frac{5}{9}} \\ \frac{4}{9} \end{bmatrix}$. So, the probability for the chain to be in state 1 after 2 transitions is $\mathbf{\frac{5}{9}}$.

   (c) Since the chain is regular, it has a steady state-vector which is the only probability vector S solving the homogeneous system $(I_2 - P)X = 0$.

$$I_2 - P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

   ($r_1 \to 3 \times r_1/2$ and $r_2 \to r_1 + r_2$). So, the homogeneous system $(I_2 - P)X = 0$ has general solution $X = t[1\ 1]^T$, $t \in \mathbb{R}$, In particular the steady state-vector (which is a solution to the homo. syst.) has the form $S = t[1\ 1]$, for some $t \in \mathbb{R}$. Since S is a probability vector, the sum of its entries is 1, therefore $t = 1/2$ and $S = \frac{1}{2}[1\ 1]^T$.

2. (a): -1; (b): -3; (c)1; (d): 0; (e):-21

3. (a): 0; (b): $\det(A) = \begin{vmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{vmatrix} = -\begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 1 & 8 & 0 \end{vmatrix} =$
$-\begin{vmatrix} 0 & 9 & 3 \\ 0 & 3 & 3 \\ 1 & 8 & 0 \end{vmatrix} = -\begin{vmatrix} 9 & 3 \\ 3 & 3 \end{vmatrix} = -18.$

4. $A^{-1} = \dfrac{1}{\det(A)}\text{adj(A)}$, where $\det(A) = -1$ and $\text{adj(A)} = [c_{ij}(A)]^T =$
$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & -1 \\ -1 & 1 & -1 \end{bmatrix}^T = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}$. Therefore, $A^{-1} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$.

5. A has determinat $\det(A) = 84$. The $(\mathbf{1}, \mathbf{3})$-entry of A^{-1} is $\frac{1}{\det(A)} c_{\mathbf{31}}(A) =$
$\frac{1}{84}(-1)^{3+1} \begin{vmatrix} -1 & 2 \\ 5 & -2 \end{vmatrix} = -\frac{8}{84} = -\frac{2}{21}$.

6. If an $n \times n$ matrix A is invertible, then $\det(\text{adj}(A)) = (\det(A))^{n-1}$. For $n = 4$, $\det(A) = -2$ and $\det(B) = 2$, we have:

   (a)

$$\begin{aligned} \det\left(\text{adj}(A) B^T A^4 (B^2)^{-1}\right) &= \det(\text{adj}(A)) \det(B^T) \det(A^4) (\det(B^2))^{-1} \\ &= (\det(A))^3 \det(B) (\det(A))^4 (\det(B))^{-2} \\ &= (\det(A))^7 \det(B)^{-1} = (-2)^7 (2)^{-1} = -64; \end{aligned}$$

   (b)

$$\begin{aligned} & \det\left(A^3 (B^2)^T ((\text{adj}(A))^{-1})^3 B^{-1}\right) \\ = \ & (\det(A))^3 (\det(B))^2 (\det(\text{adj}(A))^{-3} (\det(B)^{-1} \\ = \ & (\det(A))^3 \det(B) (\det(A))^{-9} = \det(B)(\det(A))^{-6} = 2(-2)^{-6} = 1/32. \end{aligned}$$

7. Note that if A, B and C are $n \times n$-matrices with the same rows except one, say row i, and if row i of C is a sum of row i of A and row i of B, then $\det(C) = \det(A) + \det(B)$. This holds also if rows are replaced by columns. We have:

$$\begin{aligned}
\det(C) &= \begin{vmatrix} a & b & c \\ 1 & -1 & 2 \\ d & e & f \end{vmatrix} = \begin{vmatrix} a & b & c \\ 3-2 & -2-(-2) & 1-4 \\ d & e & f \end{vmatrix} \\
&= \begin{vmatrix} a & b & c \\ 3 & -2 & 1 \\ d & e & f \end{vmatrix} + \begin{vmatrix} a & b & c \\ -2 & -(-2) & -4 \\ d & e & f \end{vmatrix} \\
&= \begin{vmatrix} a & b & c \\ 3 & -2 & 1 \\ d & e & f \end{vmatrix} - 2\begin{vmatrix} a & b & c \\ 1 & -1 & 2 \\ d & e & f \end{vmatrix} = \det(B) - 2\det(A) \\
&= 5 - 2(4) = 5 - 8 = -3
\end{aligned}$$

8. A is invertible iff $0 \neq \det(A) = c^3 - c = c(c-1)(c+1)$. So, A is invertible iff $c \in \mathbb{R} \setminus \{1, 0, -1\}$.

9. (a) $x = \dfrac{\begin{vmatrix} 4 & 2 \\ 13 & 7 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 3 & 7 \end{vmatrix}} = \dfrac{28-26}{7-6} = 2$; $y = \dfrac{\begin{vmatrix} 1 & 4 \\ 3 & 13 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 3 & 7 \end{vmatrix}} = \dfrac{13-12}{7-6} = 1$.

(b) $x = \dfrac{\begin{vmatrix} -3 & -2 & 4 \\ 0 & 3 & 1 \\ 6 & 6 & -5 \end{vmatrix}}{\begin{vmatrix} 3 & -2 & 4 \\ 5 & 3 & 1 \\ 2 & 6 & -5 \end{vmatrix}} = \dfrac{-21}{-21} = 1$; $y = \dfrac{\begin{vmatrix} 3 & -3 & 4 \\ 5 & 0 & 1 \\ 2 & 6 & -5 \end{vmatrix}}{\begin{vmatrix} 3 & -2 & 4 \\ 5 & 3 & 1 \\ 2 & 6 & -5 \end{vmatrix}} = \dfrac{21}{-21} =$

-1; $z = \dfrac{\begin{vmatrix} 3 & -2 & -3 \\ 5 & 3 & 0 \\ 2 & 6 & 6 \end{vmatrix}}{\begin{vmatrix} 3 & -2 & 4 \\ 5 & 3 & 1 \\ 2 & 6 & -5 \end{vmatrix}} = \dfrac{42}{-21} = -2$.