Practice Problems S5 (Diagonalization)

1. Let A be an $n \times n$ matrix and $0 \neq k \in \mathbb{R}$. Prove that a number λ is an eigenvalue of A iff $k\lambda$ is an eigenvalue of kA.

2. Prove that if λ is an eigenvalue of a square matrix A, then λ^5 is an eigenvalue of A^5.

3. By inspection, find the eigenvalues of

$$\text{(a)} \quad A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & -2 & 2 \\ 0 & 0 & 5 \end{bmatrix}; \quad \text{(b)} \quad B = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 2 & 5 \\ 4 & 0 & 4 \end{bmatrix}$$

4. Compute $P^{-1}AP$ and then A^n if $A = \begin{bmatrix} 6 & -5 \\ 2 & -1 \end{bmatrix}$ and $P = \begin{bmatrix} 5 & 1 \\ 2 & 1 \end{bmatrix}$

5. (Diagonalization) Find the characteristic polynomial, eigenvalues and an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix if $A = \begin{bmatrix} 3 & 1 & 1 \\ -4 & -2 & -5 \\ 2 & 2 & 5 \end{bmatrix}$.

6. Determine whether the following matrices are diagonalizable or not: (a) $A = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$; (b) $B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$; (c) $C = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$.

Solutions

1. It follows from $\det(k\lambda I_3 - k\,A) = \det\big(k(\lambda I_3 - A)\big) = k^n \det(\lambda I_3 - A)$.

2. Assume that λ is an eigenvalue of A, i.e., $AX = \lambda X$ for some nonzero vector X. It follows $A^5X = A^4(AX) = A^4(\lambda X) = \lambda A^4 X = \cdots = \lambda^5 X$. Therefore, λ^5 is an eigenvalue of A^5.

3. (a) The main diagonal entries of any triangular matrix are the eigenvalues.

   (b) A main diagonal entry in row or column with only zeros except possibly the main diagonal entry itself, is an eigenvalue.

4. $P^{-1} = \begin{bmatrix} 5 & 1 \\ 2 & 1 \end{bmatrix}^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ -2 & 5 \end{bmatrix}$. So, $P^{-1}AP = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} = \mathrm{diag}(4,1)$. It follows that $A = P\mathrm{diag}(4,1)P^{-1}$. Therefore,

$$\begin{aligned} A^n &= P\mathrm{diag}(4^n,1)P^{-1} \\ &= \frac{1}{3}\begin{bmatrix} 5 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 4^n & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -2 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 5/3\,4^n - 2/3 & -5/3\,4^n + 5/3 \\ 2/3\,4^n - 2/3 & -2/3\,4^n + 5/3 \end{bmatrix}. \end{aligned}$$

5. The characteristic polynomial of A is

$$\det(xI_3 - A) = \begin{vmatrix} x-3 & -1 & -1 \\ 4 & x+2 & 5 \\ -2 & -2 & x-5 \end{vmatrix} = (x-1)(x-2)(x-3).$$

   To simplify the computation of this determinant, subtract column 2 from column 1 and factor x-2 out of column 1 from the new determinant, then add row 1 to row 2; finally subtract column 3 from column 2. Thus A has three eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 3$. Eigenvectors: The homogeneous systems $(I_3 - A)X = 0$, $(2I_3 - A)X = 0$ and $(3I_3 - A)X = 0$ have basic solutions $X_1 = [1\ -3\ 1]^T$, $X_2 = [1\ -1\ 0]^T$ and $X_3 = [0\ -1\ 1]^T$, respectively. There are basic eigenvectors correponding to the respective eigenvalues. The matrix $P = [X_1\, X_2\, X_3] = \begin{bmatrix} 1 & 1 & 0 \\ -3 & -1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$ diagonalizes A with $P^{-1}AP = \mathrm{diag}(1,2,3)$.

6. (a) Since the 2×2 matrix A has two distinct (or simple, i.e., with multiplicity 1) eigenvalues $\lambda_1 = -5$ and $\lambda_2 = 2$, A is diagonalizable.

   (b) The matrix B has eigenvalue $\lambda = -1$ with **multiplicity** 2. The matrix B is diagonalizable if there are two basic eigenvectors corresponding $\lambda = -1$. There are basic solutions to the homogeneous system $(-I_3 - B)X = 0$. So, $X_1 = [-1\ 1, 0]$ and $X_2 = [-1, 0, 1]$ are **two basic eigenvectors** corresponding to $\lambda = -1$. The matrix B is diagolizable.

   (c) The matrix C has an eigenvalue $\lambda = 1$ of multiplicity 2. For C to be diagonalizable, there must be two basic eigenvectors corresponding to $\lambda = 1$. But the homogeneous system $(I_3 - C)X = 0$ has only one basic solution $X = [0\ 1\ 0]^T$. Therefore, C is not diagonalizable.