

Practice Problems S6 (Complex Numbers)

- Write the following complex numbers in the form $a + bi$:
(a) $\frac{3-i}{2i+5}$, (b) $(2 - 3i)^3$, (c) $\frac{1-i}{2-3i} - \frac{1+2i}{5+i}$, (d) $e^{5i\pi/3}$.
- Express the following complex numbers in polar form: (a) $(1 - \sqrt{3}i)^5$,
(b) $(\sqrt{3} - i)(2 - 2i)$, (c) $-2e^{\pi i/3}$
- Prove that $\cos(\theta_1 + \theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)$ and $\sin(\theta_1 + \theta_2) = \cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2)$.
- (a) Express the number $z = (1 - i)(-1 + \sqrt{3}i)$ in polar form and in the form $a + bi$;
(b) Find $\cos(5\pi/12)$ and $\sin(5\pi/12)$.
- Solve the following equations:
(a) $(i + z) - 3i(2 - z) = iz + 1$;
(b) $z(1 + i) = \bar{z} - (3 + 2i)$;
(c) $3x^2 + 5x + 10 = 0$;
(d) $z^2 = -15 - 8i$;
(e) $z^2 - (3 - 2i)z + (5 - i) = 0$.
- Solve the following system of linear equations:

$$\begin{cases} x + iy - iz & = 3 + i \\ -ix + 2y + iz & = 2 \\ (i - 1)x - (1 + 2i)y + 2z & = i - 1 \end{cases}.$$

- Find the inverse of $A = \begin{bmatrix} 1 & 1 - i \\ 2 + i & 3 + i \end{bmatrix}$.

8. Diagonalize the matrix $A = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$.

9. Find the 8th roots of $z = 128(-1 - \sqrt{3}i)$.

Recommended Problems:

Pages 482 - 483: 1, 2, 3 a, 4 a, b; 5 a, b, c; 6 a, b, d; 10 a, b; 11 a, b, c; 18 , 19, 23.

Solutions

1. (a) $\frac{3-i}{2i+5} = \frac{(3-i)(5-2i)}{|2i+5|^2} = \frac{13-11i}{29} = \frac{13}{29} - \frac{11}{29}i$; (b) $(2-3i)^3 = -46-9i$; (c) $e^{5\pi/3} = \cos(5\pi/3) + \sin(5\pi/3)i = \frac{1}{2} - \frac{\sqrt{3}}{2}i$.

2. (a) $(1-\sqrt{3}i)^5 = (2(1/2-\sqrt{3}/2i))^5 = (2e^{-\pi i/3})^5 = 32e^{-5\pi i/3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$; (b) $(\sqrt{3}-i)(2-2i) = 2e^{-\pi i/6}(2\sqrt{2}e^{-\pi i/4}) = 4\sqrt{2}e^{-i\pi/4-i\pi/6} = 4\sqrt{2}e^{-5\pi i/12}$; (c) $-2e^{\pi i/3} = 2(-1)e^{\pi i/3} = 2e^{\pi i}e^{\pi i/3} = 2e^{(\pi+\pi/3)i} = 2e^{4\pi i/3}$.

3. $e^{i\theta_1}e^{i\theta_2} = e^{(\theta_1+\theta_2)i}$ (Multiplication rule). It follows that

$$\begin{aligned} \left(\cos(\theta_1) + i\sin(\theta_1)\right)\left(\cos(\theta_2) + i\sin(\theta_2)\right) &= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2) \\ \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) + i\left(\cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2)\right) &= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2). \end{aligned}$$

Therefore, $\cos(\theta_1 + \theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)$ and $\sin(\theta_1 + \theta_2) = \cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2)$.

4. (a) $z = (1-i)(-1+\sqrt{3}i) = 2\sqrt{2}e^{-\pi i/4}e^{2\pi i/3} = 2\sqrt{2}e^{-\pi i/4+2\pi i/3} = 2\sqrt{2}e^{5\pi i/12}$; (b) Since $2\sqrt{2}e^{5\pi i/12} = z = (1-i)(-1+\sqrt{3}i) = (-1+\sqrt{3})+(1+\sqrt{3})i$, we have $\cos(5\pi/12) = \frac{\sqrt{2}(\sqrt{3}-1)}{4}$ and $\sin(5\pi/12) = \frac{\sqrt{2}(1+\sqrt{3})}{4}$.

5. (a) $z = 11/5 + 3i/5$; (b) $z = 2 + 3i$; (c) $x = -5/6 - i\sqrt{95}/6$ or $x = -5/6 + i\sqrt{95}/6$; (d) $z = 1 - 4i$ or $z = -1 + 4i$; (e) $z = 2 - 3i$ or $z = 1 + i$.

6. Carry the augmented matrix to its reduced row-echelon form:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1-2i & 6 \\ 0 & 1 & 1+i & 1+3i \\ 0 & 0 & 0 & 0 \end{array} \right]$$

which gives $x = 6 - (1 - 2i)t$, $y = 1 + 3i - (1 + i)t$ and $z = t$, where $t \in \mathbb{C}$. Therefore, the system has infinitely many solutions given by

$$X = \begin{bmatrix} 6 \\ 1 + 3i \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 + 2i \\ -1 - i \\ 1 \end{bmatrix}.$$

7. A has inverse $\begin{bmatrix} 1/2 - 3/2i & 1/2 + 1/2i \\ -1/2 + i & -1/2i \end{bmatrix}$.

8. A has eigenvalues $\lambda_1 = 1 - i$ and $\lambda_2 = 1 + i$ and corresponding basic eigenvectors $X_1 = [-1 \ 1]^T$ and $X_2 = [1 \ 1]^T$. Therefore, the matrix $P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$, i.e., $P^{-1}AP = \text{diag}(1 - i, 1 + i)$.

9. The complex number $z = 128(-1 - \sqrt{3}i) = 256e^{4\pi i/3} = 2^8e^{4\pi i/3}$ has 8th

roots given by $z_k = 256^{1/8}e^{\frac{4\pi/3 + 2k\pi}{8}i} = 2e^{(\pi/6 + k\pi/4)i}$, $k = 0, 1, 2, 3, \dots$.
So, $z_0 = 2e^{\pi/6i} = 2(\cos(\pi/6) + i\sin(\pi/6)) = \sqrt{3} + i$;

$$z_1 = 2e^{(\pi/6 + \pi/4)i} = 2e^{5\pi i/12} = \frac{\sqrt{2}}{2}(\sqrt{3} - 1 + i(1 + \sqrt{3}));$$

$$z_2 = 2e^{(\pi/6 + 2\pi/4)i} = 2e^{2\pi i/3} = -1 + \sqrt{3}i;$$

$$z_3 = 2e^{(\pi/6 + 3\pi/4)i} = 2e^{11\pi i/12} = \frac{\sqrt{2}}{2}(-\sqrt{3} - 1 + i(-1 + \sqrt{3}));$$

$$z_4 = 2e^{(\pi/6 + 4\pi/4)i} = 2e^{7\pi i/6} = -\sqrt{3} - i;$$

$$z_5 = 2e^{(\pi/6 + 5\pi/4)i} = 2e^{17\pi i/12} = \frac{\sqrt{2}}{2}(-\sqrt{3} + 1 - i(1 + \sqrt{3}));$$

$$z_6 = 2e^{(\pi/6 + 6\pi/4)i} = 2e^{5\pi i/3} = -1 + \sqrt{3}i;$$

$$z_7 = 2e^{(\pi/6 + 7\pi/4)i} = 2e^{23\pi i/12} = \frac{\sqrt{2}}{2}(\sqrt{3} + 1 + i(1 - \sqrt{3})).$$