Solutions

① (a) The system $\begin{cases} x + ay = 1 \\ ax + 4y = 2 \end{cases}$ has coefficient matrix $\begin{bmatrix} 1 & a \\ a & 4 \end{bmatrix}$ and augmented matrix

$$\left[\begin{array}{cc|c} 1 & a & 1 \\ a & 4 & 2 \end{array}\right]$$

(b) $$\left[\begin{array}{cc|c} \boxed{1} & a & 1 \\ a & 4 & 2 \end{array}\right] \xrightarrow{\cancel{r_2}\; r_2 - ar_1} \left[\begin{array}{cc|c} 1 & a & 1 \\ 0 & 4-a^2 & 2-a \end{array}\right]$$

Case 1: If $4-a^2 \neq 0$, i.e, $a \neq \pm 2$, then

$$\xrightarrow[\cancel{r_2}]{\frac{r_2}{4-a^2}} \left[\begin{array}{cc|c} \boxed{1} & a & 1 \\ 0 & \boxed{1} & \frac{2-a}{4-a^2} \end{array}\right] = \left[\begin{array}{cc|c} 1 & a & 1 \\ 0 & 1 & \frac{1}{2+a} \end{array}\right]$$

Back Substitution: $y = \frac{1}{2+a}$ and $x = 1 - ay$

$$x = 1 - \frac{a}{2+a} = \frac{2}{2+a}$$

In this case, the system has a unique solution $x = \frac{2}{2+a}$ and $y = \frac{1}{2+a}$.

Case 2: If $a^2-4=0$, then $a=2$ or $a=-2$

For $\boxed{a=2}$, we have:

$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & 0 \end{array}\right]$. Therefore, the system has infinitely many solutions given by $\begin{cases} x = 1-2t, & t \in \mathbb{R} \\ z = t \end{cases}$

For $\boxed{a=-2}$, we have $\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 0 & 4 \end{array}\right]$.

The system is inconsistent, i.e., has no solutions.

2. (a) $\begin{bmatrix} \boxed{1} & -1 & 3 & 5 \\ 3 & -2 & 1 & -2 \\ -1 & 1 & 1 & 3 \end{bmatrix} \xrightarrow[\cancel{r_3}\ r_3+r_1]{\cancel{r_2}\ r_2-3r_1} \begin{bmatrix} \boxed{1} & -1 & 3 & 5 \\ 0 & \boxed{1} & -8 & -17 \\ 0 & 0 & 4 & 8 \end{bmatrix}$

$\xrightarrow{\cancel{r_3}\ r_3/4} \begin{bmatrix} \boxed{1} & -1 & 3 & 5 \\ 0 & \boxed{1} & -8 & -17 \\ 0 & 0 & \boxed{1} & 2 \end{bmatrix}$ is a row-echelon matrix.

(b) This row-echelon matrix has 3 leading 1's. Therefore, the rank of A is 3.

(c) The system is equivalent to

$$\begin{cases} x_1 - x_2 + 3x_3 = 5 \\ \quad x_2 - 8x_3 = -17 \\ \quad\quad x_3 = 2 \end{cases}$$

Back-substitution: $\boxed{x_3 = 2}$

$\Longrightarrow x_2 - 16 = -17 \Rightarrow x_2 = -1$

$\Longrightarrow x_1 + 1 + 6 = 5 \Rightarrow x_1 = -2$

So, the system has exactly one solution

$x_1 = -2,\ x_3 = 2,\ x_2 = -1.$

③ (a)

$$A = \begin{bmatrix} 1 & -1 & 3 & 1 & 3 \\ -1 & -2 & 6 & 2 & -6 \\ 2 & 1 & 3 & 5 & 3 \\ 2 & -2 & 12 & 8 & 0 \end{bmatrix} \xrightarrow[\substack{\cancel{r_3}\ r_3 - 2r_1 \\ \cancel{r_4}\ r_4 - 2r_1}]{\cancel{r_2}\ r_2 + r_1} \begin{bmatrix} 1 & -1 & 3 & 1 & 3 \\ 0 & -3 & 9 & 3 & -3 \\ 0 & 3 & -3 & 3 & -3 \\ 0 & 0 & 6 & 6 & -6 \end{bmatrix}$$

$$\xrightarrow[\substack{\cancel{r_2}\ r_2/3 \\ \cancel{r_4}\ r_4/6}]{\cancel{r_3}\ r_3/(-3)} \begin{bmatrix} 1 & -1 & 3 & 1 & 3 \\ 0 & 1 & -3 & -1 & 1 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{bmatrix} \xrightarrow{\cancel{r_3}\ r_3 - r_1} \begin{bmatrix} 1 & -1 & 3 & 1 & 3 \\ 0 & 1 & -3 & -1 & 1 \\ 0 & 0 & 2 & 2 & -2 \\ 0 & 0 & 1 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow[\cancel{r_4}\ r_4 - \frac{1}{2} r_3]{\cancel{r_3}\ r_3/2} \begin{bmatrix} \boxed{1} & -1 & 3 & 1 & 3 \\ 0 & \boxed{1} & -3 & -1 & 1 \\ 0 & 0 & \boxed{1} & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ is in row-echelon form (not unique)

So, this row-echelon matrix (or any other) leads by row operations to the reduced form of A:

$$\begin{bmatrix} 1 & -1 & 3 & 1 & 3 \\ 0 & 1 & -3 & -1 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[\cancel{r_2}\ r_2 + 3r_4]{\cancel{r_1}\ r_1 + r_2} \begin{bmatrix} \boxed{1} & 0 & 0 & 0 & 4 \\ 0 & \boxed{1} & 0 & 2 & -2 \\ 0 & 0 & \boxed{1} & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ is

the reduced row-echelon matrix of A.

(b) Since the augmented matrix of the system has exactly the reduced row-echelon from part (a), the system is equivalent to

$$\begin{cases} x_1 = 4 \\ x_2 + 2x_4 = -2 \\ x_3 + x_4 = -1 \end{cases} \qquad \text{Set } x_4 = t$$

The system has infinitely many solutions

$$x_1 = 4,\ x_2 = -2-2t,\ x_3 = -1-t,\ x_4 = t$$

with $t \in \mathbb{R}$.

(4). This homogeneous system has coefficient matrix

$$\begin{bmatrix} 1 & -1 & 3 & 1 \\ -1 & -2 & 6 & 2 \\ 2 & 1 & 3 & 5 \\ 2 & -2 & 12 & 8 \end{bmatrix} \xrightarrow[\cancel{r_3}\, r_3-2r_1,\ \cancel{r_4}\, r_4-2r_1]{\cancel{r_2}\, r_2+r_1} \begin{bmatrix} 1 & -1 & 3 & 1 \\ 0 & -3 & 9 & 3 \\ 0 & 3 & -3 & 3 \\ 0 & 0 & 6 & 6 \end{bmatrix}$$

$$\xrightarrow[\cancel{r_3}\, r_3/3,\ \cancel{r_4}\, r_4/6]{\cancel{r_2}\, r_2/-3} \begin{bmatrix} 1 & -1 & 3 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{\cancel{r_3}\, r_3-r_2} \begin{bmatrix} 1 & -1 & 3 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow[\cancel{r_4}\ r_4-\frac{1}{2}r_3]{\cancel{r_3}\ r_3/2}\begin{bmatrix} \boxed{1} & -1 & 3 & 1 \\ 0 & \boxed{1} & -3 & -1 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ (row-echelon form with 3 leading 1's)

The coefficient matrix has rank 3.

Therefore, its general solution has $4-3=1$ parameter.

$$\xrightarrow[\cancel{r_1}\ r_1-3r_3]{\cancel{r_2}\ r_2+3r_3}\begin{bmatrix} \boxed{1} & -1 & 0 & -2 \\ 0 & \boxed{1} & 0 & 2 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\cancel{r_1}\ r_1+r_2} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4=t \end{array} \\ \begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 2 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

(reduced row-echelon matrix)

x_4 does not correspond to a leading 1, so, $x_4=t$

$x_3=-x_4=-t$, $x_2=-2x_4=-2t$, $x_1=0$

The general solution is

$$\begin{cases} x_1=0 \\ x_2=-2t \\ x_3=-t \\ x_4=t, \quad t\in\mathbb{R}. \end{cases}$$