Practice Problems S2

1. Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 9 & 7 \\ -1 & 0 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 3 & 2 \\ 1 & 1 & 1 \\ -1 & 4 & 1 \end{bmatrix}$.

   Compute the products AB, BA^T, BC and CB.

2. Consider the following system of linear equations:

$$\begin{cases} x_1 - 2x_2 + x_3 - 4x_4 = 1 \\ x_1 + 3x_2 + 7x_3 + 2x_4 = 2 \\ x_1 - 12x_2 - 11x_3 - 16x_4 = -1 \end{cases}.$$

   (a) Find basic solutions to the associated homogeneous system;

   (b) Find a particular solution to the system.

3. Find the general solution to the linear system $AX = B$ and specify a particular solution, where

$$A = \begin{bmatrix} 2 & 1 & -1 & -1 \\ 3 & 1 & 1 & -2 \\ -1 & -1 & 2 & 1 \\ -2 & -1 & 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 \\ -2 \\ 2 \\ 3 \end{bmatrix}.$$

   Find basic solutions and write the general solution to the associated homogeneous system as a linear combination of these basic solutions.

4. Find the inverses of the following matrices:

   (a) $\begin{bmatrix} 7 & 4 \\ 3 & 2 \end{bmatrix}$; (b) $\begin{bmatrix} 1 & 3 & 2 \\ 1 & 1 & 1 \\ -1 & 4 & 1 \end{bmatrix}$.

5. Use matrix inversion to solve the following systems of linear equations:

   (a) $\begin{cases} 7x + 4y = 2 \\ 3x + 2y = -2 \end{cases}$; (b) $\begin{cases} x + 3y + 2z = 5 \\ x + y + z = 1 \\ -x + 4y + z = 5 \end{cases}$.

6. Consider a directed graph with three vertices v_1, v_2 and v_3. Find the adjacency matrix of this graph if the edges are $v_1 \longrightarrow v_1$, $v_1 \longrightarrow v_2$, $v_2 \longrightarrow v_3$, $v_3 \longrightarrow v_2$ and $v_3 \longrightarrow v_1$. Determine the number of paths of length 5 from v_2 to v_3 and from v_3 to v_1.

Recommended Problems:

Pages 34 - 35: 1a,c; 2a,c,d,f,g; 3a,c; 4a; 5a; 8a
Pages 47 - 50: 1a,b,c,d,f.g; 2a,b; 5a; 7a,b; 8; 10; 16a; 22; 32
Pages 59 - 60: 1a,b,c; 2a,b,c,d,e,f,k; 3a,c,e; 4a; 5a,b; 6a,b.

Solutions

1. $$AB = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 1 & 9 & 7 \\ -1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -6 & -2 \\ 0 & 6 & 10 \end{bmatrix},$$

$$BA^T = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 9 & 7 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 8 \\ 6 & 30 \\ 3 & 6 \end{bmatrix},$$

$$BC = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 9 & 7 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 1 & 1 & 1 \\ -1 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 13 & 8 \\ 3 & 40 & 18 \\ -3 & 5 & 0 \end{bmatrix},$$

$$CB = \begin{bmatrix} 1 & 3 & 2 \\ 1 & 1 & 1 \\ -1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 1 & 9 & 7 \\ -1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 30 & 26 \\ 2 & 12 & 10 \\ 1 & 33 & 29 \end{bmatrix}.$$

2. The system has augmented matrix $\left[\begin{array}{cccc|c} 1 & -2 & 1 & -4 & 1 \\ 1 & 3 & 7 & 2 & 2 \\ 1 & -12 & -11 & -16 & -1 \end{array}\right]$ which can be brought by row operations to the reduced row-echelon matrix $\left[\begin{array}{cccc|c} 1 & 0 & 17/5 & -8/5 & 7/5 \\ 0 & 1 & 6/5 & 6/5 & 1/5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$.

The system has general solution

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7/5 - 17/5t + 8/5s \\ 1/5 - 6/5t - 6/5s \\ t \\ s \end{bmatrix} = \begin{bmatrix} 7/5 \\ 1/5 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -17/5 \\ -6/5 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 8/5 \\ -6/5 \\ 0 \\ 1 \end{bmatrix}.$$

$X_1 = \begin{bmatrix} 7/5 \\ 1/5 \\ 0 \\ 0 \end{bmatrix}$ is a particular solution to the system. The associated homogeneous system has general solution $X_0 = t \begin{bmatrix} -17/5 \\ -6/5 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 8/5 \\ -6/5 \\ 0 \\ 1 \end{bmatrix}$, with $X_2 = \begin{bmatrix} -17/5 \\ -6/5 \\ 1 \\ 0 \end{bmatrix}$ and $X_3 = \begin{bmatrix} 8/5 \\ -6/5 \\ 0 \\ 1 \end{bmatrix}$, or $X_4 = \begin{bmatrix} -17 \\ -6 \\ 5 \\ 0 \end{bmatrix}$ and $X_5 = \begin{bmatrix} 8 \\ -6 \\ 0 \\ 5 \end{bmatrix}$ as basic solutions.

3. The augmented matrix of the system has reduced row-echelon form

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & -4 & -9 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Therefore, the general solution is

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3-t \\ -9+4t \\ -2+t \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ -9 \\ -2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 4 \\ 1 \\ 1 \end{bmatrix}.$$

$X_1 = \begin{bmatrix} 3 \\ -9 \\ -2 \\ 0 \end{bmatrix}$ is a particular solution. The associated homogeneous system has general solution $X_0 = t \begin{bmatrix} -1 \\ 4 \\ 1 \\ 1 \end{bmatrix}$ with $\begin{bmatrix} -1 \\ 4 \\ 1 \\ 1 \end{bmatrix}$ as a basic solution.

4. (a) $\det(A) = \begin{vmatrix} 7 & 4 \\ 3 & 2 \end{vmatrix} = 7 \times 2 - 3 \times 4 = 2 \neq 0$. So, A is invertible. Its inverse $A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \frac{1}{2}\begin{bmatrix} 2 & -4 \\ -3 & 7 \end{bmatrix}$. It can be also obtained by bringing $[A|I_2]$ to the reduced row-echelon form $[I_2|A^{-1}]$.

(b) By row operations carry the matrix $\left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ -1 & 4 & 1 & 0 & 0 & 1 \end{array}\right]$ to its reduced row-echelon form ($[A|I_3] \longrightarrow [I_3|A^{-1}]$):

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 5 & 1 \\ 0 & 1 & 0 & -2 & 3 & 1 \\ 0 & 0 & 1 & 5 & -7 & -2 \end{array}\right].$$

Therefore, $A^{-1} = \begin{bmatrix} -3 & 5 & 1 \\ -2 & 3 & 1 \\ 5 & -7 & -2 \end{bmatrix}$.

5. In matrix form:

(a)

$$\begin{bmatrix} 7 & 4 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} \implies \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ 3 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$
$$= \frac{1}{2} \begin{bmatrix} 2 & -4 \\ -3 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 6 \\ -10 \end{bmatrix},$$

i.e., $x = 6$ and $y = -10$.

(b)

$$\begin{bmatrix} 1 & 3 & 2 \\ 1 & 1 & 1 \\ -1 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 5 \end{bmatrix} \implies \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 3 & 2 \\ 1 & 1 & 1 \\ -1 & 4 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 1 \\ 5 \end{bmatrix}$$
$$= \begin{bmatrix} -3 & 5 & 1 \\ -2 & 3 & 1 \\ 5 & -7 & -2 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} -5 \\ -2 \\ 8 \end{bmatrix},$$

i.e., $x = -5$, $y = -2$ and $z = 8$.

6. This directed graph has adjancency matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. The number of 5-paths (i.e., paths of length 5) from a vertex v_j to another vertex v_i is the (i, j)-entry of $A^5 = \begin{bmatrix} 5 & 3 & \mathbf{5} \\ 5 & 3 & 5 \\ 3 & \mathbf{2} & 3 \end{bmatrix}$. Therefore, there are 2 5-paths from v_2 to v_3 and 5 5-paths from v_3 to v_1.