Practice Problems S4

1. By inspection, find the determinants of the following matrices:
(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$; (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; (c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$;
(d) $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 4 \\ 2 & -4 & 6 \end{bmatrix}$; (e) $\begin{bmatrix} 1 & 0 & 4 & 9 \\ -8 & -7 & 12 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 5 & 3 \end{bmatrix}$.

2. Compute the determinants of the following matrices
(a) $A = \begin{bmatrix} -2 & 1 & 3 \\ 1 & -7 & 4 \\ -2 & 1 & 3 \end{bmatrix}$; (b) $A = \begin{bmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{bmatrix}$.

3. Find the inverse of $A = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$ using the adjoint formula.

4. Given $A = \begin{bmatrix} 3 & -1 & 2 \\ 5 & 5 & -2 \\ 1 & 2 & 3 \end{bmatrix}$, find the $(1,3)$-entry of A^{-1}.

5. If A and B are 4×4-matrices wit $\det(A) = -2$ and $\det(B) = 2$, find:
(a) $\det\left(\operatorname{adj}(A) B^T A^4 (B^2)^T\right)$; (b) $\det\left(A^3 (B^2)^T ((\operatorname{adj}(A))^{-1})^3 B^{-1}\right)$.

6. Let $A = \begin{bmatrix} a & b & c \\ 1 & -1 & 2 \\ d & e & f \end{bmatrix}$, $A = \begin{bmatrix} a & b & c \\ 3 & -2 & 1 \\ d & e & f \end{bmatrix}$ and $C = \begin{bmatrix} a & b & c \\ 1 & 0 & -3 \\ d & e & f \end{bmatrix}$
be 3×3-matrices. If $det(A) = 4$ and $\det(B) = 5$, find $\det(C)$.

7. For which values of $c \in \mathbb{R}$ is A invertible if $A = \begin{bmatrix} 1 & c & 0 \\ 2 & 0 & c \\ c & -1 & 1 \end{bmatrix}$.

8. Solve the following system by Cramer's rule:
(a) $\begin{cases} x + 2y = 4 \\ 3x + 7y = 13 \end{cases}$; (b) $\begin{cases} 3x - 2y + 4z = -3 \\ 5x + 3y + z = 0 \\ 2x + 6y - 5z = 6 \end{cases}$.

Recommended Problems:

Pages 114-115: 1. a, b, f, g, h, k, l, m, n, o, p; 5. a, b; 6, 7, 8, 9, 11, 13, 14, 15;
Pages 126-127: 1. a, c; 2. a, b, c, d; 3, 4, 6, 8, 9, 10.

Solutions

1. (a): -1; (b): -3; (c)1; (d): 0; (e):-21

2. (a): 0; (b): $\det(A) = \begin{vmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{vmatrix} = -\begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 1 & 8 & 0 \end{vmatrix} =$
$-\begin{vmatrix} 0 & 9 & 3 \\ 0 & 3 & 3 \\ 1 & 8 & 0 \end{vmatrix} = -\begin{vmatrix} 9 & 3 \\ 3 & 3 \end{vmatrix} = -18.$

3. $A^{-1} = \dfrac{1}{\det(A)}\text{adj(A)}$, where $\det(A) = -1$ and $\text{adj(A)} = [c_{ij}(A)]^T =$
$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & -1 \\ -1 & 1 & -1 \end{bmatrix}^T = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}$. Therefore, $A^{-1} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$.

4. A has determinat $\det(A) = 84$. The $(\mathbf{1}, \mathbf{3})$-entry of A^{-1} is $\frac{1}{\det(A)} c_{\mathbf{31}}(A) =$
$\frac{1}{84}(-1)^{3+1}\begin{vmatrix} -1 & 2 \\ 5 & -2 \end{vmatrix} = -\frac{8}{84} = -\frac{2}{21}$.

5. If an $n \times n$ matrix A is invertible, then $\det(\text{adj}(A)) = (\det(A))^{n-1}$. For $n = 4$, $\det(A) = -2$ and $\det(B) = 2$, we have:

   (a)

$$\begin{aligned} \det\left(\text{adj}(A)B^T A^4 (B^2)^{-1}\right) &= \det(\text{adj}(A)) \det(B^T) \det(A^4)(\det(B^2))^{-1} \\ &= (\det(A))^3 \det(B)(\det(A))^4(\det(B))^{-2} \\ &= (\det(A))^7 \det(B)^{-1} = (-2)^7(2)^{-1} = -64; \end{aligned}$$

   (b)

$$\begin{aligned} & \det\left(A^3(B^2)^T((\text{adj}(A))^{-1})^3 B^{-1}\right) \\ = \; & (\det(A))^3(\det(B))^2(\det(\text{adj}(A))^{-3}(\det(B)^{-1} \\ = \; & (\det(A))^3 \det(B)(\det(A))^{-9} = \det(B)(\det(A))^{-6} = 2(-2)^{-6} = 1/32. \end{aligned}$$

6. Note that if A, B and C are $n \times n$-matrices with the same rows except one, say row i, and if row i of C is a sum of row i of A and row i of B,

then $\det(C) = \det(A) + \det(B)$. This holds also if rows are replaced by columns. We have:

$$\begin{aligned}
\det(C) &= \begin{vmatrix} a & b & c \\ 1 & -1 & 2 \\ d & e & f \end{vmatrix} = \begin{vmatrix} a & b & c \\ 3-2 & -2-(-2) & 1-4 \\ d & e & f \end{vmatrix} \\
&= \begin{vmatrix} a & b & c \\ 3 & -2 & 1 \\ d & e & f \end{vmatrix} + \begin{vmatrix} a & b & c \\ -2 & -(-2) & -4 \\ d & e & f \end{vmatrix} \\
&= \begin{vmatrix} a & b & c \\ 3 & -2 & 1 \\ d & e & f \end{vmatrix} - 2\begin{vmatrix} a & b & c \\ 1 & -1 & 2 \\ d & e & f \end{vmatrix} = \det(B) - 2\det(A) \\
&= 5 - 2(4) = 5 - 8 = -3
\end{aligned}$$

7. A is invertible iff $0 \neq \det(A) = c^3 - c = c(c-1)(c+1)$. So, A is invertible iff $c \in \mathbb{R} \setminus \{1, 0, -1\}$.

8. (a) $x = \dfrac{\begin{vmatrix} 4 & 2 \\ 13 & 7 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 3 & 7 \end{vmatrix}} = \dfrac{28-26}{7-6} = 2$; $y = \dfrac{\begin{vmatrix} 1 & 4 \\ 3 & 13 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 3 & 7 \end{vmatrix}} = \dfrac{13-12}{7-6} = 1.$

   (b) $x = \dfrac{\begin{vmatrix} -3 & -2 & 4 \\ 0 & 3 & 1 \\ 6 & 6 & -5 \end{vmatrix}}{\begin{vmatrix} 3 & -2 & 4 \\ 5 & 3 & 1 \\ 2 & 6 & -5 \end{vmatrix}} = \dfrac{-21}{-21} = 1$; $y = \dfrac{\begin{vmatrix} 3 & -3 & 4 \\ 5 & 0 & 1 \\ 2 & 6 & -5 \end{vmatrix}}{\begin{vmatrix} 3 & -2 & 4 \\ 5 & 3 & 1 \\ 2 & 6 & -5 \end{vmatrix}} = \dfrac{21}{-21} =$

   -1; $z = \dfrac{\begin{vmatrix} 3 & -2 & -3 \\ 5 & 3 & 0 \\ 2 & 6 & 6 \end{vmatrix}}{\begin{vmatrix} 3 & -2 & 4 \\ 5 & 3 & 1 \\ 2 & 6 & -5 \end{vmatrix}} = \dfrac{42}{-21} = -2.$