SOLUTIONS
MATH 211 PRACTICE PROBLEMS

1. (a) $x_1 = -1 - 2r - 3s - t$, $x_2 = r$, $x_3 = 2 + s - t$, $x_4 = s$, $x_5 = t$ and $x_6 = 3$.

   (b) $x_1 = -1 - 7s - t$, $x_2 = 0$, $x_3 = s$, $x_4 = 3 + 3t$, $x_5 = t$ and $x_6 = 0$.

2. (a) rank is 3 (b) rank is 2

   (c) rank is 2 if $a = 5$, and rank is 3 if $a \neq 5$.

   (d) rank is 2 if $a = 1$, and rank is 3 if $a \neq 1$.

3. The rank r of the augmented matrix satisfies $r \leq 5$ because there are 5 equations. Hence there are $7 - r \geq 2$ parameters by Theorem 3 §1.2, and so there is more than one solution.

4. The system may have no solution. So assume it is consistent. The rank of the augmented matrix is $r = 4$, and there are $n = 4$ variables, so there are $n - r = 0$ parameters. In other words, the solution is unique.

5. The solutions are the coordinates of points lying on all three planes. If the three planes are all parallel, there is no solution unless they all coincide, in which case there are infinitely many solutions (any point on the common plane). If two of the planes are not parallel, they intersect in a line. If this line is not parallel to the third plane, it meets it in a unique solution; otherwise the line is either in the third plane (infinitely many solutions) or it does not meet the third plane (no solution).

6. If $A \to B$ by a row interchange, the reverse is the same interchange.

   If $A \to B$ by multiplying row k by $c \neq 0$, the reverse is multiplying row k by $1/c$.

   If $A \to B$ by adding c times row i to row j, the reverse is subtracting c times row i from row j.

7. $$\begin{bmatrix} b_i + c_i \\ c_i + a_i \\ a_i + b_i \end{bmatrix} \to \begin{bmatrix} b_i + c_i \\ a_i - b_i \\ a_i + b_i \end{bmatrix} \to \begin{bmatrix} b_i + c_i \\ 2a_i \\ a_i + b_i \end{bmatrix} \to \begin{bmatrix} b_i + c_i \\ a_i \\ b_i \end{bmatrix} \to \begin{bmatrix} c_i \\ a_i \\ b_i \end{bmatrix} \to \begin{bmatrix} a_i \\ c_i \\ b_i \end{bmatrix} \to \begin{bmatrix} a_i \\ b_i \\ c_i \end{bmatrix}$$

8. $x^2 + y^2 - 3x - \frac{1}{3}y - \frac{4}{3} = 0$.

9. $f(x) = 1 - 2x + 3x^2$.

10. (a) If $a = 2$, $x_1 = t$, $x_2 = t$ and $x_3 = t$; if $a \neq 2$, $x_1 = x_2 = x_3 = 0$.

    (b) If $a = -3$, $x_1 = 9t$, $x_2 = -5t$ and $x_3 = t$; if $a \neq -3$, $x_1 = x_2 = x_3 = 0$.

    (c) If $a \neq 0$ and $a \neq -1$, then $x_1 = x_2 = x_3 = 0$;

    If $a = 0$ then $x_3 = 0$, $x_1 = -t$ and $x_2 = t$;

    If $a = -1$ then $x_1 = 3t$, $x_2 = -2t$ and $x_3 = t$.

    (d) If $a \neq 1$ and $a \neq -1$, then $x_1 = x_2 = x_3 = 0$;

    If $a = 1$ then $x_3 = 0$, $x_1 = -t$ and $x_2 = t$;

    If $a = -1$ then $x_1 = t$, $x_2 = 0$ and $x_3 = t$.

11. (a) If $xA + yB + zC = 0$ then equating corresponding entries gives

$$\begin{array}{rcrcrcl} x & & & + & z & = & 0 \\ -x & + & 3y & + & z & = & 0 \\ & & y & + & 2z & = & 0 \end{array}$$

The only solution is $x = y = z = 0$, so $\{A, B, C\}$ is linearly independent.

(b) $3\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 3 \end{bmatrix} - 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ so these matrices are not linearly independent.

12. Given points (x_1, y_1) and (x_2, y_2), the line with equation $ax + by + c = 0$ passes through these points if $ax_1 + by_1 + c = 0$ and $ax_2 + by_2 + c = 0$. These are two homogeneous equations in the three variables a, b and c, and so has a nontrivial solution by Theorem 1 § 1.3. The line corresponding to this solution will contain both points.

13. Given points (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3), the plane with equation $ax+by+cz+d = 0$ passes through these points if $ax_1 + by_1 + cz_1 + d = 0$, $ax_1 + by_1 + cz_2 + d = 0$ and $ax_1 + by_1 + cz_3 + d = 0$. These are three homogeneous equations in the four variables a, b, c and d, and so has a nontrivial solution by Theorem 1 § 1.3. The plane corresponding to this solution will contain all three points.

14. $[x\ y\ z]^T = [1 + 3t \quad 1 - 5t \quad t]^T$.

15. $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

16. $[x\ y\ z]^T = [\frac{15}{2} \quad -4 \quad -\frac{1}{2}]^T$.

17. $[x_1\ x_2\ x_3\ x_4\ x_5\ ]^T = [2 + s - 5t \quad s \quad -3 - 2t \quad 2t \quad t]^T$.

18. $\begin{bmatrix} 1 & -1 & 2 & a \\ 2 & -1 & 3 & b \\ -1 & 2 & -3 & c \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 & a \\ 0 & 1 & -1 & b-2a \\ 0 & 1 & -1 & c+a \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & b-a \\ 0 & 1 & -1 & b-2a \\ 0 & 0 & 0 & c+3a-b \end{bmatrix}$.

Hence if $c \neq b - 3a$ there is no solution.

If $c = b - 3a$ there are infinitely many solutions: $[x\ y\ z]^T = [b - a - t \quad b - 2a + t \quad t]^T$.

19. $\begin{bmatrix} 1 & -1 & 2 & a \\ 2 & 1 & -1 & 3 \\ 1 & 5 & -8 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 & a \\ 0 & 3 & -5 & 3-2a \\ 0 & 6 & -10 & 1-a \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 & a \\ 0 & 3 & -5 & 3-2a \\ 0 & 0 & 0 & 3a-5 \end{bmatrix}$.

Hence if $a \neq \frac{5}{3}$ there is no solution.

If $a = \frac{5}{3}$ there are infinitely many solutions.

20. False. If $\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 3 \end{bmatrix}$ is the augmented matrix, there is no row of zeros but infinitely many solutions: $x = 2 + t$, $y = 3 + t$, $z = t$.

21. The reduction of the augmented matrix to reduced row-echelon form is:

$$\begin{bmatrix} 1 & -1 & 2 & 2 & 3 & -4 \\ -2 & 3 & -6 & -3 & -11 & 11 \\ -1 & 2 & -4 & 1 & -8 & 7 \\ 0 & 1 & -2 & 3 & -5 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & -2 & -1 \\ 0 & 1 & -2 & 0 & -5 & 3 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence $[x_1\ x_2\ x_3\ x_4\ x_5]^T = [-1+2t \quad 3+2s+5t \quad s \quad 0 \quad t]^T$.

22. $\begin{bmatrix} 1 & 0 & 1 & 0 & -2 & -2 \\ 0 & 1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 & 6 \end{bmatrix}$.

23. (a) The reduced form for A is $\begin{bmatrix} 1 & 0 & 8 & -5 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ so $X = \begin{bmatrix} -8s+5t \\ -3s+3t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -8 \\ -3 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 5 \\ 3 \\ 0 \\ 1 \end{bmatrix}$.

(b) The reduced form for A is $\begin{bmatrix} 1 & -3 & 0 & 4 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ so $X = \begin{bmatrix} 3s-4t \\ s \\ -2t \\ t \\ 0 \end{bmatrix} = s\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}$.

24. Here $A = [A_1\ A_2\ A_3\ A_4] = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 \end{bmatrix}$, and B is a linear combination of the columns $A_1,\ A_2,\ A_3$, and A_4 if and only if $AX = B$ for some X, that is if the system $AX = B$ has a solution. Moreover, if $X = [x_1\ x_2\ x_3\ x_4]^T$ then $B = x_1A_1 + x_2A_2 + x_3A_3 + x_4A_4$.

(a) If $B = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ the augmented matrix for the system $AX = B$ is $[A \mid B] = \left[\begin{array}{cccc|c} 2 & 3 & 4 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 & -1 \end{array}\right]$,

and this has reduced form $\left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 2 \\ 0 & 1 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$. Hence there are solutions $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, in fact the general solution is $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2+s-2t \\ -1-2s+t \\ s \\ t \end{bmatrix}$. So $B = x_1A_1 + x_2A_2 + x_3A_3 + x_4A_4$ for any values of s and t (check this). Taking $s = t = 0$ we have a particular linear combination $B = 2A_1 - A_2$.

(b) If $B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ the augmented matrix is $[A \mid B] = \left[\begin{array}{cccc|c} 2 & 3 & 4 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \end{array}\right]$ and this has reduced form $\left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$ and so has no solution. So this B is not a linear combination of $A_1,\ A_2,\ A_3$, and A_4.

25. Let $A = [A_1 \ \ A_2 \ \ \cdots \ \ A_n]$ where A_i is column i of A for each i, and write $X = [x_1 \ \ x_2 \ \ \cdots \ \ x_n]^T$ and $Y = [y_1 \ \ y_2 \ \ \cdots \ \ y_n]^T$. Then

$$AX = x_1A_1 + x_2A_2 + \cdots x_nA_n \text{ and } AY = y_1A_1 + y_2A_2 + \cdots + y_nA_n.$$

Now observe that

$$X + Y = [x_1 + y_1 \ \ x_2 + y_2 \ \ \cdots \ \ x_n + y_n]^T.$$

Hence

$$\begin{aligned} A(X+Y) &= (x_1+y_1)A_1 + (x_2+y_2)A_2 + \cdots + (x_n+y_n)A_n \\ &= (x_1A_1 + y_1A_1) + (x_2A_2 + y_2A_2) + \cdots + (x_nA_n + y_nA_n) \\ &= (x_1A_1 + x_2A_2 + \cdots x_nA_n) + (y_1A_1 + y_2A_2 + \cdots + y_nA_n) \\ &= AX \ + \ AY. \end{aligned}$$

26. Let $X = \begin{bmatrix} x \\ y \end{bmatrix}$. Much as in Example 14, $T(X) = \begin{bmatrix} y \\ -x \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = AX$ where $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

27. Let $X = \begin{bmatrix} x \\ y \end{bmatrix}$. Then $T(X) = \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = AX$ where $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

28. The $(3,2)$-entry is the dot product of row 3 of A with column of B, that is $7 \cdot 4 + (-3) \cdot 1 + 4 \cdot (-2) = 17$.

29. (a) $A = \begin{bmatrix} \frac{7}{5} & \frac{2}{5} \end{bmatrix}$; (b) $A = \frac{1}{5} \begin{bmatrix} 3 & -2 \\ 0 & 6 \end{bmatrix}$.

30. (a) $A = 4B$; (b) $A = -\frac{5}{8}B^T$.

31. Every 1×3 matrix A can be written in the form $A = \begin{bmatrix} a & b & c \end{bmatrix}$ for some scalars a, b and c. Hence

$$A = \begin{bmatrix} a & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & c \end{bmatrix} = a\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.$$

If $A = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is a 3×1 matrix then $A = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ in the same way.

32. If $A = -A$ then adding A to both sides gives $2A = A + (-A) = 0$. As $2 \neq 0$, this means $A = 0$.

33. Suppose A is symmetric, that is $A^T = A$. Then $(cA)^T = c\,A^T = cA$ using Theorem 4 §1.1, that is cA is symmetric.

34. $(-A)^T = ((-1)A)^T = (-1)\,A^T = -A^T$ by Theorem 4 §1.1.

35. If A and B are symmetric then $A^T = A$ and $B^T = B$. Hence Theorem 4 §1.1 and Exercise 6 give $(A - B)^T = (A + (-B))^T = A^T + (-B)^T = A^T + (-B^T) = A^T - B^T = A - B$. Hence $A - B$ is also symmetric.

36. (a) If A is skew symmetric and 2×2, write $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ for some scalars a, b, c and d. Since $A^T = -A$ we have $\begin{bmatrix} a & c \\ b & d \end{bmatrix} = -\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$. Equating entries gives $a = -a$, $c = -b$ and $d = -d$. Hence $A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$ for some scalar b.

(b) If A and B are skew-symmetric then $A^T = -A$, and $B^T = -B$. Hence $(A+B)^T = A^T + B^T = -A + (-B) = -(A+B)$, so $A+B$ is skew symmetric. Similarly, $(cA)^T = c\,A^T = c(-A) = -(cA)$ shows that cA is skew-symmetric for any scalar c.

37. As in the Hint: $\frac{1}{2}(A+A^T)+\frac{1}{2}(A-A^T) = \frac{1}{2}A+\frac{1}{2}A^T+\frac{1}{2}A-\frac{1}{2}A^T = A$. If we take $S = \frac{1}{2}(A+A^T)$ and $W = \frac{1}{2}(A-A^T)$ then $A = S + W$ and:

$S^T = \frac{1}{2}(A^T + A^{TT}) = \frac{1}{2}(A^T + A) = S$, so S is symmetric,

$W^T = \frac{1}{2}(A^T - A^{TT}) = \frac{1}{2}(A^T - A) = -W$, so W is skew-symmetric.

38. (a) $3AB + 4BA$.

(b) $AB - BA$

(c) $CA^2C - ABCB$

(d) 0

39. If $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$ then $0 = A^2 = \begin{bmatrix} a^2+b^2 & ab+bd \\ ab+bd & b^2+d^2 \end{bmatrix}$. Hence $a^2+b^2 = 0 = b^2+d^2$ and it follows (since a, b and d are real numbers) that $a = b = d = 0$. Hence $A = 0$.

40. $AA^T = \begin{bmatrix} a^2+b^2+c^2 & aa_1+bb_1+cc_1 \\ aa_1+bb_1+cc_1 & a_1^2+b_1^2+c_1^2 \end{bmatrix}$. So $AA^T = 0$ means $a^2+b^2+c^2 = 0 = a_1^2+b_1^2+c_1^2$. Since A has real entries, this means $a = b = c = 0$ and $a_1 = b_1 = c_1 = 0$; that is $A = 0$.

41. Write $B = AA^T$. Using Theorem 1 §1.4, we get $B^T = (AA^T)^T = A^{TT}A^T = AA^T = B$. Hence B is a symmetric matrix.

42. We are given that $CA = AC$ and $CB = BC$. Hence

$$C(2A - 3B) = 2CA - 3CB = 2AC - 3BC = (2A - 3B)C.$$

Hence $2A - 3B$ commutes with C too.

43. We use Theorem 3 §1.5.

$$A^T - 3\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} = \left\{A^T - 3\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}\right\}^{-1-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} = \frac{1}{1}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.$$

Hence $A^T = 3\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 5 & -1 \end{bmatrix}$, so $A = A^{TT} = \begin{bmatrix} 4 & 5 \\ -1 & -1 \end{bmatrix}$.

44. We use Theorem 3 §1.5.

$$A - 2I = (A - 2I)^{-1-1} = \left\{A^{-1}\begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix}\right\}^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix}^{-1} A^{-1-1} = \frac{1}{1}\begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix} A.$$

Thus $2I = A - \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix} A = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix} \right\} A = \begin{bmatrix} -2 & 1 \\ -1 & 1 \end{bmatrix} A$. Hence

$A = \begin{bmatrix} -2 & 1 \\ -1 & 1 \end{bmatrix}^{-1} 2I = 2 \left\{ \frac{1}{-1} \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix} \right\} = 2 \begin{bmatrix} -1 & 1 \\ -1 & 2 \end{bmatrix}$.

45. If it happens that A^{-1} exists, then $AX = 0$ gives $A^{-1}AX = A^{-1}0$, that is $IX = 0$, that is $X = 0$. This is contrary to our assumption, so A^{-1} does not exist.

46. U is invertible because $det U = 15 + 28 \neq 0$. So $AU = 0$ gives $A = AI = AUU^{-1} = 0U^{-1} = 0$.

47. Since B is invertible, we have $A = (AB)B^{-1}$, and this is invertible by Theorem 3 §1.5 because both AB and B^{-1} are invertible.

48. Since $AB = cI$, multiplying by $\frac{1}{c}$ gives $A(\frac{1}{c}B) = I$. Hence $(\frac{1}{c}B)A = I$ by Corollary 2 of Theorem 5 §1.5. Multiplying by c gives $BA = cI$, as required. The result is false if $c = 0$, even for 2×2 matrices: If $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ then $AB = 0$ but $BA \neq 0$.

49. Write the equation $A^3 - 2A^2 + 5A + 6I = 0$ in the form $A(A^2 - 2A + 5) = -6I$ so $A \cdot \frac{1}{6}() = I$. Similarly $\frac{1}{6}(-A^2 + 2A - 5) \cdot A = I$. These equations show that A is invertible and that $A^{-1} = \frac{1}{6}(-A^2 + 2A - 5)$.

50. Showing that $A^{-1} = A$ is the same as showing that $A^2 = I$. Since we have $E^2 = E$, we have

$$A^2 = (I - 2E)^2 = (I - 2E)(I - 2E) = I^2 - 2IE - 2EI + 4E^2 = I - 2E - 2E + 4E = I.$$

51. Using the matrix inversion algorithm (or otherwise), we have $A^{-1} = \begin{bmatrix} -10 & 4 & 3 \\ -8 & 3 & 2 \\ -3 & 1 & 1 \end{bmatrix}$.

52. If row 1 of A consists of zeros then $YA = 0$ where $Y = [1\ 0\ 0\ \cdots\ 0]$. Hence if A^{-1} exists then $Y = YI = YAA^{-1} = 0A^{-1} = 0$, a contradiction. So A^{-1} does not exist.

53. To solve $AX = B$, left multiply both sides by A^{-1} to get $A^{-1}AX = A^{-1}B$, that is $X = A^{-1}B$. So every solution (if there is one) must equal $A^{-1}B$. But $X = A^{-1}B$ is indeed a solution because $AX = A(A^{-1}B) = IB = B$.

54. $det \begin{bmatrix} a+2x & b+2y & c+2z \\ x+p & y+q & z+r \\ 3p & 3q & 3r \end{bmatrix} = 3\, det \begin{bmatrix} a+2x & b+2y & c+2z \\ x+p & y+q & z+r \\ p & q & r \end{bmatrix}$

$= 3\, det \begin{bmatrix} a+2x & b+2y & c+2z \\ x & y & z \\ p & q & r \end{bmatrix} = 3\, det \begin{bmatrix} a & b & c \\ x & y & z \\ p & q & r \end{bmatrix} = -3\, det \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix} = -15.$

55. Compute $det \begin{bmatrix} 1 & c & 0 \\ 2 & 0 & c \\ c & -1 & 1 \end{bmatrix} = det \begin{bmatrix} 1 & 0 & 0 \\ 2 & -2c & c \\ c & -1-c^2 & 1 \end{bmatrix} = det \begin{bmatrix} -2c & c \\ -1-c^2 & 1 \end{bmatrix} = c(c^2 - 1) = c(c-1)(c+1)$.

Hence the matrix is invertible if $c \neq 0$, 1 and -1.

56. Using the matrix inversion algorithm we have $\begin{bmatrix} 1 & -1 & -2 \\ -1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 2 & 1 \\ -2 & -4 & -1 \\ 1 & 3 & 1 \end{bmatrix}$.

The equations are $AX = B$ in matrix form where $B = [3\ 0\ 1]^T$, so the solution is

$$X = A^{-1}B = \begin{bmatrix} 1 & 2 & 1 \\ -2 & -4 & -1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -7 \\ 4 \end{bmatrix}.$$

57. Since $A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$ is invertible, it can be carried to the identity by row operations. These row operations generate elementary matrices as follows:

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \to E_1A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \to E_2(E_1A) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \to E_3(E_2E_1A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

where (by Lemma 1 §2.5) $E_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$ and $E_3 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$. Hence $(E_3E_2E_1)A = I$ so (using Lemma 2 §2.5) we have

$$A = (E_3E_2E_1)^{-1} = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

You can check this by direct matrix multiplication.

58. It is enough to find the matrix A of T, and $A = [T(E_1)\ \ T(E_2)]$ by Theorem 1 where $E_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $E_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Write $P = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for convenience, so $T(P) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $T(Q) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Now observe that $E_1 = -P + 2Q$ and $E_2 = P - Q$. Since T is linear we obtain

$$T(E_1) = -T(P) + 2T(Q) = \begin{bmatrix} 5 \\ 1 \end{bmatrix} \quad \text{and} \quad T(E_2) = T(P) - T(Q) = \begin{bmatrix} -3 \\ 0 \end{bmatrix}.$$

Hence $A = [T(E_1)\ \ T(E_2)] = \begin{bmatrix} 5 & -3 \\ 1 & 0 \end{bmatrix}$, so $T(X) = AX = \begin{bmatrix} 5x - 3y \\ x \end{bmatrix}$ is the desired formula.

59. Rotation through $\pi/2$ has matrix $R_{\pi/2} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, and reflection in the line $y = x$ has matrix $Q_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Thus the transformation in question has matrix $Q_1R_{\pi/2} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, which is reflection in the X-axis.

60. We have $T\left([x\ y\ 0]^T\right) = [x\ y\ 0]^T$ for any x and y because $[x\ y\ 0]^T$lies in the x-y plane. The reflection of a vector $[0\ 0\ z]^T$ in the x-y plane is $[0\ 0\ -z]^T$, that is $T\left([0\ 0\ z]^T\right) = [0\ 0\ -z]^T$. Hence:

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = T\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} + T = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -z \end{bmatrix} = \begin{bmatrix} x \\ y \\ -z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

for all x, y and z, so T is multiplication by the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. In particular T is linear.

61. Reflection in the line $y = -3x$ has matrix $\frac{1}{10}\begin{bmatrix} -8 & -6 \\ -6 & 8 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -4 & -3 \\ -3 & 4 \end{bmatrix}$ by Example 10. Hence the reflection of $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ in $y = -3x$ is $\frac{1}{5}\begin{bmatrix} -4 & -3 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 \\ -18 \end{bmatrix}$.

62. (a) The matrix is $A = \left[T\begin{bmatrix} 1 \\ 0 \end{bmatrix} \;\; T\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]$. We are given $T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$, and also $T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$. Now observe that $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ so, as T is linear, $T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = T\begin{bmatrix} 1 \\ 1 \end{bmatrix} - T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 9 \end{bmatrix}$. Hence the matrix of T is $A = \left[T\begin{bmatrix} 1 \\ 0 \end{bmatrix} \;\; T\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right] = \begin{bmatrix} 2 & 3 \\ 9 & -2 \end{bmatrix}$. This gives $T\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 9 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x+3y \\ 9x-2y \end{bmatrix}$.

(b) The matrix of T^{-1} is $A^{-1} = \frac{1}{-31}\begin{bmatrix} -2 & -3 \\ -9 & 2 \end{bmatrix} = \frac{1}{31}\begin{bmatrix} 2 & 3 \\ 9 & -2 \end{bmatrix}$. Hence $T^{-1}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = A^{-1}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \frac{1}{31}\begin{bmatrix} 10 \\ 14 \end{bmatrix}$.

63. $detB = det\begin{bmatrix} 2x & a+2p & p-3x \\ 2y & b+2q & q-3y \\ 2z & c+2r & r-3z \end{bmatrix} = 2\,det\begin{bmatrix} x & a+2p & p-3x \\ y & b+2q & q-3y \\ z & c+2r & r-3z \end{bmatrix} = 2\,det\begin{bmatrix} x & a+2p & p \\ y & b+2q & q \\ z & c+2r & r \end{bmatrix} =$
$2\,det\begin{bmatrix} x & a & p \\ y & b & q \\ z & c & r \end{bmatrix} = 2\,det\begin{bmatrix} x & y & z \\ a & b & c \\ p & q & r \end{bmatrix} = -2\,det\begin{bmatrix} a & b & c \\ x & y & z \\ p & q & r \end{bmatrix}$

$= 2\,det\begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix} = 2\,detA = 6$. Finally $det(-2B^{-1}) = (-2)^3\frac{1}{detB} = -\frac{8}{6} = -\frac{4}{3}$.

64. If $A^2 = -I$ then $(detA)^2 = det\,A^2 = det(-I) = (-1)^3 = -1$. This is impossible as $detA$ is a real number.

65. $det\begin{bmatrix} p+x & q+y & r+z \\ a+x & b+y & c+z \\ a+p & b+q & c+r \end{bmatrix} = det\begin{bmatrix} p-a & q-b & r-c \\ a+x & b+y & c+z \\ a+p & b+q & c+r \end{bmatrix} = det\begin{bmatrix} 2p & 2q & 2r \\ a+x & b+y & c+z \\ a+p & b+q & c+r \end{bmatrix}$

$= 2\,det\begin{bmatrix} p & q & r \\ a+x & b+y & c+z \\ a+p & b+q & c+r \end{bmatrix} = 2\,det\begin{bmatrix} p & q & r \\ a+x & b+y & c+z \\ a & b & c \end{bmatrix} = 2\,det\begin{bmatrix} p & q & r \\ x & y & z \\ a & b & c \end{bmatrix}$

$= -2\,det\begin{bmatrix} a & b & c \\ x & y & z \\ p & q & r \end{bmatrix} = 2\,det\begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}$.

66. $det\begin{bmatrix} 1 & a & p & q \\ x & 1 & b & r \\ x^2 & x & 1 & c \\ x^3 & x^2 & x & 1 \end{bmatrix} = det\begin{bmatrix} 1-ax & a & p & q \\ 0 & 1 & b & r \\ 0 & x & 1 & c \\ 0 & x^2 & x & 1 \end{bmatrix} = (1-ax)det\begin{bmatrix} 1 & b & r \\ x & 1 & c \\ x^2 & x & 1 \end{bmatrix}$

$= (1-ax)det\begin{bmatrix} 1-bx & b & r \\ 0 & 1 & c \\ 0 & x & 1 \end{bmatrix} = (1-ax)(1-bx)det\begin{bmatrix} 1 & c \\ x & 1 \end{bmatrix} = (1-ax)(1-bx)(1-cx).$

67. (a) $\begin{bmatrix} a & 3-a & a+1 \\ b & 3-b & b+1 \\ c & 3-c & c+1 \end{bmatrix} \to \begin{bmatrix} a & 3 & 1 \\ b & 3 & 1 \\ c & 3 & 1 \end{bmatrix}$ so A has zero determinant.

(b) $\begin{bmatrix} a & b & c \\ a+b & 2b & c+b \\ 3 & 3 & 3 \end{bmatrix} \to \begin{bmatrix} a & b & c \\ b & b & b \\ 3 & 3 & 3 \end{bmatrix}$ so A has zero determinant.

68. $detB = det\begin{bmatrix} a+c & 2c \\ b+d & 2d \end{bmatrix} = 2\,det\begin{bmatrix} a & c \\ b & d \end{bmatrix} = 2\,det(A^T) = 2\,detA = 4$. Hence

$det(A^2B^TA^{-1}) = (detA)^2\,detB\,\dfrac{1}{detA} = 8.$

69. $det\begin{bmatrix} x-1 & 2 & 3 \\ 2 & -3 & x-2 \\ -2 & x & -2 \end{bmatrix} = det\begin{bmatrix} x-1 & x-1 & x-1 \\ 2 & -3 & x-2 \\ -2 & x & -2 \end{bmatrix} = det\begin{bmatrix} x-1 & 0 & 0 \\ 2 & -5 & x-4 \\ -2 & x+2 & 0 \end{bmatrix}$

$= -(x-1)(x+2)(x-4)$. Hence the determinant is zero if $x = 1$, -2 or 4.

70. If $A^2 = 3A$ then $(detA)^2 = det(A^2) = det(3A) = 3^4 detA$. Hence either $detA = 0$ or $detA = 3^4 = 81$.

71. $det\begin{bmatrix} 3 & -3 & 0 \\ c+5 & -5 & 3a \\ d-2 & 2 & 3b \end{bmatrix} = 3det\begin{bmatrix} 3 & -3 & 0 \\ c+5 & -5 & a \\ d-2 & 2 & b \end{bmatrix} = 3\,det\begin{bmatrix} 3 & 0 & 0 \\ c+5 & c & a \\ d-2 & d & b \end{bmatrix} = 3\left\{3\,det\begin{bmatrix} c & a \\ d & b \end{bmatrix}\right\}$

$= -9det\begin{bmatrix} a & c \\ b & d \end{bmatrix} = -9\,detA = 27.$

72. If $AB = -BA$ then $detA\,detB = det(AB) = det(-BA) = (-1)^n det(BA) = -detA\,detB$. Since $detA$ and $detB$ are numbers, this gives $2\,detA\,detB = 0$. Hence $detA = 0$ or $detB = 0$.

73. We have $A^{-1} = \dfrac{1}{detA} adjA = \frac{1}{2} adjA$, so $adjA = 2\,detA$. Using the fact that A^{-1} is of size 4×4, this gives

$$det(A^{-1} - 6\,adjA) = det(15A^{-1} - 12A^{-1}) = det(3\,A^{-1}) = 3^4 det(A^{-1}) = 81\tfrac{1}{2} = \tfrac{81}{2}.$$

74. (a) $detA = 2c^2 - c - 1 = (2c+1)(c-1)$ so A is invertible unless $c = -\frac{1}{2}$ or $c = 1$. If $c \neq -\frac{1}{2}$ and $c \neq 1$ then $A^{-1} = \dfrac{1}{2c^2-c-1}\begin{bmatrix} 2c+1 & -2c-1 & 0 \\ -2+c & c & -c+1 \\ -1-c^2 & c+c^2 & c^2-c \end{bmatrix}$.

(b) Here $det A = 2$ for all values of c, so A is invertible and $A^{-1} = \frac{1}{2}\begin{bmatrix} 8-c^2 & -c & c^2-6 \\ c & 1 & -c \\ c^2-10 & c & 8-c^2 \end{bmatrix}$.

75. In each case we use Theorem 3 §2.2 twice.

(a) $det\begin{bmatrix} A & X & Y \\ 0 & B & Z \\ 0 & 0 & C \end{bmatrix} = det A \, det\begin{bmatrix} B & Z \\ 0 & C \end{bmatrix} = det A\,(det B \, det C) = -6.$

(b) $det\begin{bmatrix} A & X & 0 \\ 0 & B & 0 \\ Y & Z & C \end{bmatrix} = det\begin{bmatrix} A & X \\ 0 & B \end{bmatrix} det C = (det A \, det B) det C = -6.$

76. As in the Hint: $\begin{bmatrix} 0 & I_2 \\ I_3 & 0 \end{bmatrix}\begin{bmatrix} 0 & B \\ A & X \end{bmatrix} = \begin{bmatrix} A & X \\ 0 & B \end{bmatrix}$. Hence $det\begin{bmatrix} 0 & I_2 \\ I_3 & 0 \end{bmatrix} det\begin{bmatrix} 0 & B \\ A & X \end{bmatrix} = det A \, det B$. But $det\begin{bmatrix} 0 & I_2 \\ I_3 & 0 \end{bmatrix} = 1$ by direct calculation, and the result follows.

77. $c_A(x) = det\begin{bmatrix} x & -2 \\ -2 & x+3 \end{bmatrix} = (x-1)(x+4)$ so $\lambda_1 = 1$ and $\lambda_2 = -4$ are the eigenvalues.

$\lambda_1 I - A = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$, so $X_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector.

$\lambda_2 I - A = \begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$, so $X_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ is an eigenvector.

Hence $P = [X_1 \; X_2] = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ has $P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix}$.

78. $c_A(x) = det\begin{bmatrix} x+2 & -1 \\ -4 & x-1 \end{bmatrix} = (x-2)(x+3)$ so $\lambda_1 = 2$ and $\lambda_2 = -3$ are the eigenvalues.

$\lambda_1 I - A = \begin{bmatrix} 4 & -1 \\ -4 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 4 & -1 \\ 0 & 0 \end{bmatrix}$, so $X_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ is an eigenvector.

$\lambda_2 I - A = \begin{bmatrix} -1 & -1 \\ -4 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, so $X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector.

Hence $P = [X_1 \; X_2] = \begin{bmatrix} 1 & -1 \\ 4 & 1 \end{bmatrix}$ has $P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$.

79. SOLUTION 1. $c_A(x) = det\begin{bmatrix} x & -1 \\ 1 & x-2 \end{bmatrix} = (x-1)^2$, so $\lambda_1 = 1$ is the only eigenvalue (of multiplicity 2). Here $\lambda_1 I - A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$, so $X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the only basic eigenvector. Hence A is not diagonalizable by Theorem 5 §2.3.

SOLUTION 2. If A were diagonalizable, there would exist an invertible matrix P such that $P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{bmatrix} = I$. Hence $A = PIP^{-1} = I$, a contradiction. So A is not diagonalizable.

80. Let $A^k = 0$ where $k \geq 1$. If λ is an eigenvalue of A then $AX = \lambda X$ for some eigenvector $X \neq 0$. Hence $A^2X = A(\lambda X) = \lambda(AX) = \lambda(\lambda X) = \lambda^2 X$. Similarly, $A^3X = \lambda^3 X$, and eventually $A^kX = \lambda^k X$. Since $A^k = 0$, this gives $\lambda^k X = 0$, and so $\lambda^k = 0$ because $X \neq 0$. Thus $\lambda = 0$.

81. Since A is diagonalizable, there exists an invertible matrix P such that $P^{-1}AP = D$ is diagonal. But the diagonal entries of D are just the eigenvalues of A in some order, so $D = 0$ by hypothesis. Hence $P^{-1}AP = 0$, so that $A = P0P^{-1} = 0$.

82. If λ is an eigenvalue of A then $AX = \lambda X$ for some eigenvector $X \neq 0$. Then (as above) $A^2X = \lambda^2 X$. But $A^2 = A$ so this gives $\lambda^2 X = AX = \lambda X$. Hence $(\lambda^2 - \lambda)X = 0$, whence $\lambda^2 = \lambda$ because $X \neq 0$. This means that $\lambda = 0$ or $\lambda = 1$.

83. Let $P^{-1}AP = D = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$ where the λ_i are the eigenvalues of A. Since we are assuming that $\lambda_i^2 = \lambda_i$ for each i, we have $D^2 = diag(\lambda_1^2, \lambda_2^2, \cdots, \lambda_n^2) = diag(\lambda_1, \lambda_2, \cdots, \lambda_n) = D$. Hence, $A^2 = (PDP^{-1})^2 = PD^2P^{-1} = PDP^{-1} = A$.

84. Let $P^{-1}AP = D$ where D is diagonal. Then $(PAP^{-1})^2 = D^2$, that is $PA^2P^{-1} = D^2$. Since D^2 is also diagonal, this shows that A^2 is also diagonalizable (with the same P).

85. Let $P^{-1}AP = D$ where D is diagonal. Then $(PAP^{-1})^T = D^T = D$, that is $(P^{-1})^T A^T P^T = D$. If we write $Q = P^T$, then $Q^{-1} = (P^T)^{-1} = (P^{-1})^T$, so we have $Q^{-1}A^TQ = D$. This shows that A^T is diagonalizable.

86. $c_A(x) = det \begin{bmatrix} x-2 & -1 & -1 \\ -1 & x-1 & 1 \\ -1 & -1 & x-2 \end{bmatrix} = det \begin{bmatrix} x-3 & -1 & -1 \\ 0 & x-1 & 1 \\ x-3 & -1 & x-2 \end{bmatrix}$ after adding column 3 to column 1. It follows that $c_A(x) = (x-1)^2(x-3)$, so $\lambda_1 = 1$ is an eigenvalue of multiplicity 2. The basic eigenvectors corresponding to λ_1 are the basic solutions to the equations$(\lambda_1 I - A)X = 0$. The reduction of the augmented matrix is $[\lambda_1 I - A \;\; 0] = \begin{bmatrix} -1 & -1 & -1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & -1 & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Hence $X_2 = \left[\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}\right]$ is the only basic eigenvector corresponding to λ_1, so A is not diagonalizable by Theorem 5 §2.3.

87. Here $c_A(x) = det \begin{bmatrix} x & -1 & -1 \\ -1 & x & -1 \\ -2 & 0 & x \end{bmatrix} = det \begin{bmatrix} x-2 & -1 & -1 \\ x-2 & x & -1 \\ x-2 & 0 & x \end{bmatrix}$ after adding columns 2 and 3 to column 1. Hence $c_A(x) = (x+1)^2(x-2)$, so $\lambda_1 = -1$ is an eigenvalue of multiplicity 2. But the basic eigenvectors corresponding to λ_1 are the basic solutions to the homogeneous system $(\lambda_1 I - A)X = 0$. The augmented matrix is reduced as follows: $\begin{bmatrix} -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \\ -2 & 0 & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, so there is only one parameter, and so only one basic solution. Since the multiplicity of λ_1 is 2, this shows that A is not diagonalizable by Theorem 5 §2.3.

88. Let $P^{-1}AP = D = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$ where the λ_i are the eigenvalues of A. Since we are assuming that $\lambda_i \geq 0$ for each i, $D_0 = diag(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \cdots \sqrt{\lambda_n})$ is a real diagonal matrix such that $D_0^2 = D$. Put $B = PD_0P^{-1}$. Then $B^2 = (PD_0P^{-1})^2 = PD_0^2P^{-1} = PDP^{-1} = A$.

89. Let $P^{-1}AP = D = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$ where the λ_i are the eigenvalues of A. We are assuming that each $\lambda_i = \lambda$, so $P^{-1}AP = diag(\lambda, \lambda, \cdots, \lambda) = \lambda I$. Hence $A = P(\lambda I)P^{-1} = \lambda I$.

90. Let $P^{-1}AP = D = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$ where the λ_i are the eigenvalues of A. Then $A = PDP^{-1}$ so $detA = detP\, detD\, \dfrac{1}{detP} = detD = \lambda_1\lambda_2 \cdots \lambda_n$.

91. Here $A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$, so $c_A(x) = det(xI - A) = det \begin{bmatrix} x-1 & -4 \\ -1 & x-1 \end{bmatrix} = x^2 - 2x - 3 = (x-3)(x+1)$. Hence the eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = -1$. The eigenvectors corresponding to an eigenvalue λ are the nonzero solutions to $(\lambda I - A)X = 0$. In our case:

$$\lambda_1 = 3: \ (3I - A) = \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix} \text{ so an eigenvector is } X_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

$$\lambda_2 = -1: \ (-I - A) = \begin{bmatrix} -2 & -4 \\ -1 & -2 \end{bmatrix} \text{ so an eigenvector is } X_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Hence $P = [X_1 \ X_2] = \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix}$ is a diagonalizing matrix for A, that is $P^{-1}AP = D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$. (Verify by showing $AP = PD$.) Hence $A = PDP^{-1}$ so

$$\begin{aligned} A^n = (PDP^{-1})^n = PD^nP^{-1} &= \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3^n & 0 \\ 0 & (-1)^n \end{bmatrix} \left(\tfrac{1}{4} \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} \right) \\ &= \begin{bmatrix} \frac{1}{2}(-1)^n + \frac{1}{2}3^n & 3^n - (-1)^n \\ \frac{1}{4}3^n - \frac{1}{4}(-1)^n & \frac{1}{2}(-1)^n + \frac{1}{2}3^n \end{bmatrix}. \end{aligned}$$

This is the desired formula for A^n.

92. Suppose that the recurrence is $x_{n+2} = ax_n + bx_{n+1}$ where a and b are fixed numbers. The idea here is to calculate the columns $V_n = \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix}$ rather than the x_n themselves. The reason is that

$$V_{n+1} = \begin{bmatrix} x_{n+1} \\ x_{n+2} \end{bmatrix} = \begin{bmatrix} x_{n+1} \\ ax_n + nx_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a & b \end{bmatrix} \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix} = AV_n \text{ for each } n.$$

Thus the sequence V_0, V_1, V_2, ... is a dynamical system, so we get an exact formula for the V_n if A is diagonalizable. In our case $a = 2$ and $b = 1$, so $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$. Thus $c_A(x) = det(xI - A) = det \begin{bmatrix} x & -1 \\ -2 & x-1 \end{bmatrix} = x^2 - x - 2 = (x-2)(x+1)$.

Hence the eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = -1$. The eigenvectors corresponding to any eigenvalue λ are the nonzero solutions X to $(\lambda I - A)X = 0$. In our case:

$$\lambda_1 = 2 \text{ so } (\lambda_1 I - A) = (2I - A) = \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix} \text{ so one eigenvector is } X_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

$\lambda_2 = -1$ so $(\lambda_2 I - A) = (-I - A) = \begin{bmatrix} -1 & -1 \\ -2 & -2 \end{bmatrix}$ and an eigenvector is $X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Thus $P = [X_1 \;\; X_2] =$ is a diagonalizing matrix for A, that is $P^{-1}AP = D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$. This giver $A = PDP^{-1}$.

Now $V_n = A^n V_0$ where $V_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ since we are assuming that $x_0 = - = x_1$. Hence we have

$$V_n = (PDP^{-1})^n V_0 = PD^n(P^{-1}V_0) = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2^n & 0 \\ 0 & (-1)^n \end{bmatrix} \left(\frac{1}{3} \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right) = \frac{1}{3} \begin{bmatrix} 2^{n+1} + (-1)^n \\ 2^{n+2} + (-1)^{n+1} \end{bmatrix}$$

Since the top entry of V_n is x_n we get $x_n = \frac{1}{3}(2^{n+1} + (-1)^n)$ as the required exact formula. You can check it against the first few values $x_0 = 1$, $x_1 = 1$, $x_2 = 3$, etc.

93. $detA = (1+i) + i^2 = i$, so $A^{-1} = \frac{1}{i} \begin{bmatrix} 1+i & -i \\ i & 1 \end{bmatrix} = (-i) \begin{bmatrix} 1+i & -i \\ i & 1 \end{bmatrix} = \begin{bmatrix} 1-i & -1 \\ 1 & -i \end{bmatrix}$.

94. If we write $z = 2-3i$, then the quadratic $(x-z)(x-\bar{z}) = x^2-(z+\bar{z})x+z\bar{z}$ has real coefficients. Indeed, $z + \bar{z} = 4$ is twice the real part of z, and $z\bar{z} = 13 = |z|^2$ is the square of the absolute value of z. So the required polynomial is $x^2 - 4x + 13$. The other root is $\bar{z} = 2 + 3i$.

95. $w^2 - 6w + 13 = (5 - 12i) - (18 - 12i) + 13 = 0$. The other root is $\bar{w} = 3 + 2i$.

96. $\bar{z} = \overline{(1+i)^n + (1-i)^n} = \overline{(1+i)^n} + \overline{(1-i)^n} = \overline{(1+i)}^n + \overline{(1-i)}^n = (1-i)^n + (1+i)^n = z$. Hence z is real.

97. $z(\frac{1}{|z|^2}\bar{z}) = \frac{1}{|z|^2}(z\bar{z}) = \frac{1}{|z|^2}(|z|^2) = 1$. The result follows.

98. If zw is real and $z \neq 0$, then $\frac{w}{\bar{z}} = \frac{zw}{z\bar{z}} = \frac{zw}{|z|^2}$ is real. So take $r = \frac{zw}{|z|^2}$.

99. By the Hint:
$$\begin{aligned} |z+w|^2 + |z-w|^2 &= (z+w)(\bar{z}+\bar{w}) + (z-w)(\bar{z}-\bar{w}) \\ &= (z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w}) + (z\bar{z} - z\bar{w} - w\bar{z} + w\bar{w}) \\ &= 2(z\bar{z} + w\bar{w}) \\ &= 2(|z|^2 + |w|^2). \end{aligned}$$

100. Let $\vec{p} = [2 \;\; -1 \;\; 5]^T$ and $\vec{q} = [3 \; 0 \; 4]^T$ be the position vectors of the points. The position vector of the point $\frac{1}{5}$ the way from $\vec{p}$ to $\vec{q}$ is $\vec{t} = \vec{p} + \frac{1}{5}(\vec{q} - \vec{p}) = \frac{4}{5}\vec{p} + \frac{1}{5}\vec{q} = \frac{1}{5}[11 \;\; -4 \;\; 24]^T$.

101. Let $\vec{p} = [1 \; 2 \; 3]^T$ and $\vec{q} = [8 \;\; -2 \;\; 0]^T$ be the position vectors of the points. The position vector of the two points are $\vec{t_1} = \vec{p} + \frac{1}{3}(\vec{q} - \vec{p}) = \frac{2}{3}\vec{p} + \frac{1}{3}\vec{q} = \frac{2}{3}[5 \; 1 \; 3]^T$ and $\vec{t_2} = \vec{p} + \frac{2}{3}(\vec{q} - \vec{p}) = \frac{1}{3}\vec{p} + \frac{2}{3}\vec{q} = \frac{1}{3}[17 \;\; -2 \;\; 3]^T$.

102. $\overrightarrow{AE} = \overrightarrow{AB} + \overrightarrow{BE}$ and also $\overrightarrow{AE} = \overrightarrow{AC} + \overrightarrow{CE}$. Hence

$$\overrightarrow{AE} = \tfrac{1}{2}(\overrightarrow{AE} + \overrightarrow{AE}) = \tfrac{1}{2}[(\overrightarrow{AB} + \overrightarrow{BE}) + (\overrightarrow{AC} + \overrightarrow{CE})] = \tfrac{1}{2}[\overrightarrow{AB} + \overrightarrow{AC}]$$

because $\overrightarrow{BE} = -\overrightarrow{CE}$ (since E is the midpoint of BC).

103. The diagonals are

$$[1\ 1\ 1]^T - [0\ 0\ 0]^T = [1\ 1\ 1]^T, \qquad [0\ 1\ 1]^T - [1\ 0\ 0]^T = [-1\ 1\ 1]^T,$$
$$[1\ 0\ 1]^T - [0\ 1\ 0]^T = [1\ -1\ 1]^T, \qquad [1\ 1\ 0]^T - [0\ 0\ 1]^T = [1\ 1\ -1]^T.$$

No pair is orthogonal as the dot products are all nonzero.

104. (a) Let $\vec{v}_1 = proj_{\vec{d}}(\vec{v}) = (\frac{\vec{v} \bullet \vec{d}}{|\vec{d}|^2})\vec{d} = \frac{1}{2}\vec{d} = \frac{1}{2}[1\ 2\ 1]^T$. Then $\vec{v}_2 = \vec{v} - \vec{v}_1 = \frac{1}{2}[5\ -4\ 3]^T$. A check on the arithmetic is that $\vec{v}_2 \bullet \vec{d} = 0$ should hold.
(b) Let $\vec{v}_1 = proj_{\vec{d}}(\vec{v}) = (\frac{\vec{v} \bullet \vec{d}}{|\vec{d}|^2})\vec{d} = \frac{1}{2}\vec{d} = \frac{1}{2}[3\ 0\ -7]^T$. Then $\vec{v}_2 = \vec{v} - \vec{v}_1 = \frac{1}{2}[7\ 2\ 3]^T$.

105. We have $\|\vec{v} + \vec{w}\|^2 = \|\vec{v}\|^2 + 2\,\vec{v} \bullet \vec{w} + \|\vec{w}\|^2$. Hence the condition $\|\vec{v}\|^2 + \|\vec{w}\|^2 = \|\vec{v} + \vec{w}\|^2$ gives $2\,\vec{v} \bullet \vec{w} = 0$, that is $\vec{v} \bullet \vec{w} = 0$. This means that $\vec{v}$ and $\vec{w}$ are orthogonal.

106. Here $\vec{d} = [-5\ 0\ 2]^T$ from the given line, so the equation is $[x\ y\ z]^T = [3\ -1\ 2]^T + t[-5\ 0\ 2]^T$.

107. Now $\vec{d} = \overrightarrow{P_1P_2} = [1\ 1\ 1]^T$, so the line is $[x\ y\ z]^T = [1\ 0\ -2]^T + t[1\ 1\ 1]^T$.

108. Every point on the line has the form $[x\ y\ z]^T = [2+t \quad -1-t \quad 3-4t]^T$. This point lies on the plane if $3(2+t) + (-1-t) - 2(3-4t) = 4$, which gives $t = \frac{1}{2}$. Hence the point is$[x\ y\ z]^T = [\frac{5}{2} \quad -\frac{3}{2} \quad 1]^T$.

109. The point $P_0(3,-1,0)$ is in the plane, so the vector $\vec{v} = \overrightarrow{P_0P} = [-2\ 2\ -2]^T$ is in the plane. Since $\vec{d} = [1\ 1\ -1]^T$ is also in the plane, a normal is $\vec{n} = \vec{v} \times \vec{d} = [0\ -4\ -4]^T$. Thus the lane has equation $y + z = k$ for some number k. Since $P_0(3,-1,0)$ lies in the plane, $k = -1$, and the equation is $y + z = -1$.

110. The normal $\vec{n} = [1\ 1\ -2]^T$ will serve as direction vector of the line (it is perpendicular to the plane). As $P(1,-1,0)$ is in the line, the equation is $[x\ y\ z]^T = [1\ -1\ 0]^T + t[1\ 1\ -2]^T$.

111. As the planes are parallel, the normal $\vec{n} = [4\ -3\ 1]^T$ of the given plane will serve for the new one. So the new plane has equation $4x - 3y + z = d$ for some scalar d. Since $P_0(2,3,-1)$ lies in the new plane, $d = -2$ and the equation is $4x - 3y + z = -2$.

112. Write $P_0 = P_0(1,2,0)$ and $\vec{v} = \overrightarrow{P_0P} = [-1\ -1\ 2]^T$. Compute $\vec{v}_1 = proj_{\vec{d}}(\vec{v}) = \frac{1}{6}[2\ -1\ 1]^T$. The position vector of Q is then $\vec{q} = \vec{p}_0 + \vec{v}_1 = \frac{1}{6}[8\ 11\ 1]^T$.

113. A normal to the plane is $\vec{n} = [5\ -7\ 2]^T$. Choose any point in the plane, say $P_0(1,0,-1)$, and write $\vec{v} = \overrightarrow{P_0P} = [0\ 0\ 3]^T$. Then the shortest distance from P to the plane is $\|proj_{\vec{n}}(\vec{v})\| = \left\|(\frac{\vec{v} \bullet \vec{n}}{\|\vec{n}\|^2})\vec{n}\right\| = \frac{|\vec{v} \bullet \vec{n}|}{\|\vec{n}\|} = \frac{6}{\sqrt{78}}$.

114. Write $P_0 = P_0(1,-1,0)$, $\vec{v} = \overrightarrow{P_0P} = [0\ 1\ 2]^T$ and $\vec{d} = [2\ 1\ 1]^T$. Compute $\vec{v}_1 = proj_{\vec{d}}(\vec{v}) = \frac{1}{2}\vec{d}$. Then the shortest distance is $\|\vec{v} - \vec{v}_1\| = \left\|\frac{1}{2}[-2\ 1\ 3]^T\right\| = \frac{1}{2}\sqrt{14}$.

115. The plane in question has equation $2x - 3y + 2z = d$ for some number d (using the same normal as the given plane). As it contains the point $P_0(1,-1,0)$, we obtain $d = 5$, so the equation is $2x - 3y + 2z = 5$. This does not pass through the origin.

116. We have $\overrightarrow{AB} = [-1\ \ -1\ \ -1]^T$, $\overrightarrow{AC} = [0\ 1\ \ -4]^T$ and $\overrightarrow{BC} = [1\ 2\ \ -3]^T$. Hence $\overrightarrow{AB} \bullet \overrightarrow{BC} = 0$ so the angle at B is a right angle. If θ is the internal angle at C then, since $\overrightarrow{CA} = [0\ \ -1\ \ 4]^T$ and $\overrightarrow{CB} = [-1\ \ -2\ \ 3]^T$, we have $cos\theta = \dfrac{\overrightarrow{CA} \bullet \overrightarrow{CB}}{\left\|\overrightarrow{CA}\right\| \left\|\overrightarrow{CB}\right\|} = \dfrac{14}{\sqrt{17}\sqrt{14}} = \dfrac{\sqrt{14}\sqrt{17}}{17}$.

117. $\overrightarrow{AB} = [-1\ \ 1\ \ 0]^T$ and $\overrightarrow{AC} = [-1\ \ 0\ \ 1]^T$. The area of the triangle is half the area of the parallelogram, that is $\frac{1}{2}\left\|\overrightarrow{AB} \times \overrightarrow{AC}\right\| = \frac{1}{2}\left\|[1\ 1\ 1]^T\right\| = \frac{\sqrt{3}}{2}$.