MATHEMATICS 221 L04/12/22 FALL 2003 QUIZ 3 SOLUTION
Thursday, October 23, 2000 at 11:00

1. Let $A = \begin{bmatrix} 5 & 3 \\ -6 & -4 \end{bmatrix}$.

(a) Is A diagonaliable? If A is diagonalizable find an invertible matrix P so that $P^{-1}AP = D$ where D is a diagonal matrix.

Solution:

$c_A(x) = \begin{vmatrix} x-5 & -3 \\ 6 & x+4 \end{vmatrix} = (x-5)(x+4) + 18 = x^2 - x - 2 = (x+1)(x-2)$.

Solve $c_A(x) = 0$, we get the eigenvalues of A are -1 and 2. Since A is 2×2 and A has two different eigenvalues, A is diagonalizable.

To find the eigenvectors of A corresponding to the eigenvalue -1, we solve $(-I - A)X = 0$.

$$\begin{bmatrix} -6 & -3 & 0 \\ 6 & 3 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, the eigenvectors of A corresponding to the eigenvalue -1 are

$X = t[1, -2]^T$ where t is any non-zero number.

To find the eigenvectors of A corresponding to the eigenvalue 2, we solve $(2I - A)X = 0$.

$$\begin{bmatrix} -3 & -3 & 0 \\ 6 & 6 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, the eigenvectors of A corresponding to the eigenvalue 2 are

$X = t[-1, 1]^T$ where t is any non-zero number.

Put $P = \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$. Then $P^{-1}AP = D$.

(b) Compute A^8.

Solution:

From $P^{-1}AP = D$, we have $A = PDP^{-1}$, and so

$$\begin{aligned} A^8 &= PD^8P^{-1} \\ &= \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} (-1)^8 & 0 \\ 0 & 2^8 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 256 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -512 & -256 \end{bmatrix} \\ &= \begin{bmatrix} 511 & 255 \\ -510 & -254 \end{bmatrix} \end{aligned}$$

2. Let $A = \begin{bmatrix} 2 & 1 & 2 \\ -3 & -1 & -1 \\ 5 & 2 & 1 \end{bmatrix}$.

(a) Find $adj A$.

Solution:

$c_{11}(A) = \begin{vmatrix} -1 & -1 \\ 2 & 1 \end{vmatrix} = 1$, $c_{21}(A) = -\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 3$, $c_{31}(A) = \begin{vmatrix} 1 & 2 \\ -1 & -1 \end{vmatrix} = 1$

$c_{12}(A) = -\begin{vmatrix} -3 & -1 \\ 5 & 1 \end{vmatrix} = -2$, $c_{22}(A) = \begin{vmatrix} 2 & 2 \\ 5 & 1 \end{vmatrix} = -8$, $c_{32}(A) = -\begin{vmatrix} 2 & 2 \\ -3 & -1 \end{vmatrix} = -4$

$c_{13}(A) = \begin{vmatrix} -3 & -1 \\ 5 & 2 \end{vmatrix} = -1$, $c_{23}(A) = -\begin{vmatrix} 2 & 1 \\ 5 & 2 \end{vmatrix} = 1$, $c_{33}(A) = \begin{vmatrix} 2 & 1 \\ -3 & -1 \end{vmatrix} = 1$

Thus, $adj A = \begin{bmatrix} 1 & 3 & 1 \\ -2 & -8 & -4 \\ -1 & 1 & 1 \end{bmatrix}$.

(b) Compute $A.adj A$.

Solution:

$$A.adj A = \begin{bmatrix} 2 & 1 & 2 \\ -3 & -1 & -1 \\ 5 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ -2 & -8 & -4 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

(c) Find $\det A$.

Solution: From (b) and $A.adj A = (\det A) I$, we get $\det A = -2$.

(d) Can we use Cramer's Rule for the following system? Explain. Do not solve the system.

$$\begin{array}{rcrcrcl} 2x & + & y & + & 2z & = & 1 \\ -3x & - & y & - & z & = & 0 \\ 5x & + & 2y & + & z & = & 1 \end{array}$$

Solution: From (b) and $\det A = -2 \neq 0$, we can use Cramer's Rule to solve the above system.

MATHEMATICS 221 L04/12/22 FALL 2003 QUIZ 3 SOLUTION
Thursday, October 23, 2000 at 16:00

1. Let $A = \begin{bmatrix} 5 & 6 \\ -3 & -4 \end{bmatrix}$.

(a) Is A diagonaliable? If A is diagonalizable find an invertible matrix P so that $P^{-1}AP = D$ where D is a diagonal matrix.

Solution:

$c_A(x) = \begin{vmatrix} x-5 & -6 \\ 3 & x+4 \end{vmatrix} = (x-5)(x+4) + 18 = x^2 - x - 2 = (x+1)(x-2)$.

Solve $c_A(x) = 0$, we get the eigenvalues of A are -1 and 2. Since A is 2×2 and A has two different eigenvalues, A is diagonalizable.

To find the eigenvectors of A corresponding to the eigenvalue -1, we solve $(-I - A)X = 0$.

$$\begin{bmatrix} -6 & -6 & 0 \\ 3 & 3 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, the eigenvectors of A corresponding to the eigenvalue -1 are

$X = t[-1, 1]^T$ where t is any non-zero number.

To find the eigenvectors of A corresponding to the eigenvalue 2, we solve $(2I - A)X = 0$.

$$\begin{bmatrix} -3 & -6 & 0 \\ 3 & 6 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, the eigenvectors of A corresponding to the eigenvalue 2 are

$X = t[-2, 1]^T$ where t is any non-zero number.

Put $P = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$. Then $P^{-1}AP = D$.

(b) Compute A^8.

Solution:

From $P^{-1}AP = D$, we have $A = PDP^{-1}$, and so

$$\begin{aligned} A^8 &= PD^8P^{-1} \\ &= \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (-1)^8 & 0 \\ 0 & 2^8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 256 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -256 & -256 \end{bmatrix} \\ &= \begin{bmatrix} 511 & 510 \\ -255 & -254 \end{bmatrix} \end{aligned}$$

2. Let $A = \begin{bmatrix} 3 & 5 & 1 \\ 1 & 2 & -1 \\ -1 & 0 & 1 \end{bmatrix}$.

(a) Find $adj A$.

Solution:

$$c_{11}(A) = \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = 2,\ c_{21}(A) = -\begin{vmatrix} 5 & 1 \\ 0 & 1 \end{vmatrix} = -5,\ c_{31}(A) = \begin{vmatrix} 5 & 1 \\ 2 & -1 \end{vmatrix} = -7$$

$$c_{12}(A) = -\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = 0,\ c_{22}(A) = \begin{vmatrix} 3 & 1 \\ -1 & 1 \end{vmatrix} = 4,\ c_{32}(A) = -\begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix} = 4$$

$$c_{13}(A) = \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} = 2,\ c_{23}(A) = -\begin{vmatrix} 3 & 5 \\ -1 & 0 \end{vmatrix} = -5,\ c_{33}(A) = \begin{vmatrix} 3 & 5 \\ 1 & 2 \end{vmatrix} = 1$$

Thus, $adj A = \begin{bmatrix} 2 & -5 & -7 \\ 0 & 4 & 4 \\ 2 & -5 & 1 \end{bmatrix}$.

(b) Compute $A.adj A$.

Solution:

$$A.adj A = \begin{bmatrix} 3 & 5 & 1 \\ 1 & 2 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -5 & -7 \\ 0 & 4 & 4 \\ 2 & -5 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

(c) Find $\det A$.

Solution: From (b) and $A.adj A = (\det A)\, I$, we get $\det A = 8$.

(d) Can we use Cramer's Rule for the following system? Explain. Do not solve the system.

$$\begin{array}{rcrcrcr} 3x & + & 5y & + & z & = & 0 \\ x & + & 2y & - & z & = & 1 \\ -x & & & + & z & = & -1 \end{array}$$

Solution: From (b) and $\det A = 8 \neq 0$, we can use Cramer's Rule to solve the above system.