MATHEMATICS 221 L04/12/22 FALL 2003 QUIZ 4 SOLUTIONS
Thursday, November 20, 2003 at 16:00

1. Given $P\,(1,1,3)$ and $Q\,(0,-3,5)$. Find the coordinates of the point A that is $\frac{1}{5}$ the way from P to Q.
Solution:

$$\begin{aligned}\overrightarrow{OA} &= \overrightarrow{OP}+\frac{1}{5}\overrightarrow{PQ} = [1,1,3]^T+\frac{1}{5}[-1,-4,2]^T \\ &= \frac{1}{5}\left([5,5,15]^T+[-1,-4,2]^T\right) \\ &= \frac{1}{5}[4,1,17]^T=\left[\tfrac{4}{5},\tfrac{1}{5},\tfrac{17}{5}\right]^T\end{aligned}$$

Thus, the coordinates of A is $A\left(\frac{4}{5},\frac{1}{5},\frac{17}{5}\right)$.

2. Let A,B,C and D be the vertices in order of a parallelogram $ABCD$. Given $A\,(1,-1,2)$, $C\,(2,1,0)$ and the midpoint $M\,(1,0,-3)$ of the side AB, find $\overrightarrow{BD}$.
Solution:

$$\begin{aligned}\overrightarrow{BD} &= \overrightarrow{BA}+\overrightarrow{AD}=2\overrightarrow{MA}+\overrightarrow{AC}+\overrightarrow{CD} && \text{because} \\ &= 2\overrightarrow{MA}+\overrightarrow{AC}+2\overrightarrow{MA}=4\overrightarrow{MA}+\overrightarrow{AC} && \text{because } \overrightarrow{CD}=\overrightarrow{BA}=2\overrightarrow{MA} \\ &= 4\left(\overrightarrow{OA}-\overrightarrow{OM}\right)+\overrightarrow{OC}-\overrightarrow{OA}=3\overrightarrow{OA}-4\overrightarrow{OM}+\overrightarrow{OC} \\ &= 3[1,-1,2]^T-4[1,0,-3]^T+[2,1,0]^T \\ &= [3,-3,6]^T-[4,0,-12]^T+[2,1,0]^T \\ &= [3-4+2,-3-0+1,6+12+0]^T \\ &= [1,-2,18]^T\end{aligned}$$

3. Find an equation of the plane passing through $A\,(3,1,2)$, $B\,(5,-1,2)$ and $C\,(-4,2,0)$.
Solution:
A point of the plane is $C\,(-4,2,0)$ and a normal of the plane is

$$\begin{aligned}\overrightarrow{n} &= \frac{1}{4}\overrightarrow{AB}\times\overrightarrow{AC}=\frac{1}{4}[2,-2,0]^T\times[-7,1,-2]^T \\ &= \frac{1}{4}\left[\begin{vmatrix}-2 & 0\\ 1 & -2\end{vmatrix}, -\begin{vmatrix}2 & 0\\ -7 & -2\end{vmatrix}, \begin{vmatrix}2 & -2\\ -7 & 1\end{vmatrix}\right]^T \\ &= \frac{1}{4}[4,4,-12]^T=[1,1,-3]^T\end{aligned}$$

Thus, an equation of the plane is $(x-(-4))+(y-2)-3\,(z-0)=0$ or $x+y-3z=-6$.

4. Show that for all vectors $\overrightarrow{u}$ and $\overrightarrow{v}$, $\|\overrightarrow{u}+\overrightarrow{v}\|^2+\|\overrightarrow{u}-\overrightarrow{v}\|^2=2\left(\|\overrightarrow{u}\|^2+\|\overrightarrow{v}\|^2\right)$.
Solution:

$$\begin{aligned}&\|\overrightarrow{u}+\overrightarrow{v}\|^2+\|\overrightarrow{u}-\overrightarrow{v}\|^2 \\ &=(\overrightarrow{u}+\overrightarrow{v})\bullet(\overrightarrow{u}+\overrightarrow{v})+(\overrightarrow{u}-\overrightarrow{v})\bullet(\overrightarrow{u}-\overrightarrow{v}) \\ &=\overrightarrow{u}\bullet\overrightarrow{u}+\overrightarrow{u}\bullet\overrightarrow{v}+\overrightarrow{v}\bullet\overrightarrow{u}+\overrightarrow{v}\bullet\overrightarrow{v}+\overrightarrow{u}\bullet\overrightarrow{u}-\overrightarrow{u}\bullet\overrightarrow{v}-\overrightarrow{v}\bullet\overrightarrow{u}+\overrightarrow{v}\bullet\overrightarrow{v} \\ &=2\overrightarrow{u}\bullet\overrightarrow{u}+2\overrightarrow{v}\bullet\overrightarrow{v}=2\,(\overrightarrow{u}\bullet\overrightarrow{u}+\overrightarrow{v}\bullet\overrightarrow{v})=2\left(\|\overrightarrow{u}\|^2+\|\overrightarrow{v}\|^2\right)\end{aligned}$$

Thursday, November 20, 2003 at 11:00

1. Find the coordinates of the two points that trisect the line segment connecting $A\,(-5,1,3)$ and $B\,(-4,-5,0)$.
Solution:
Let the two points that trisect the line segment connecting $A\,(-5,1,3)$ and $B\,(-4,-5,0)$ be P and Q. Then
$\overrightarrow{OP}=\overrightarrow{OA}+\overrightarrow{AP}=\overrightarrow{OA}+\frac{1}{3}\overrightarrow{AB}=[-5,1,3]^T+\frac{1}{3}[1,-6,-3]^T=\left[\frac{-14}{3},-1,2\right]^T$ and
$\overrightarrow{OQ}=\overrightarrow{OA}+\overrightarrow{AQ}=\overrightarrow{OA}+\frac{2}{3}\overrightarrow{AB}=[-5,1,3]^T+\frac{2}{3}[1,-6,-3]^T=\left[\frac{-13}{3},-3,1\right]^T$. Thus these two points are $P\left(\frac{-14}{3},-1,2\right)$ and $Q\left(\frac{-13}{3},-3,1\right)$.
2. Let A and B be the end-points of a diameter of a circle. If C is any other point on the circle, show that the line segment AC and BC are perpendicular.
Solution:
Let O be the centre of the circle. Then

$$\begin{aligned}\overrightarrow{AC}\bullet\overrightarrow{BC} &= \left(\overrightarrow{AO}+\overrightarrow{OC}\right)\bullet\left(\overrightarrow{BO}+\overrightarrow{OC}\right)\\ &= \left(\overrightarrow{AO}+\overrightarrow{OC}\right)\bullet\left(-\overrightarrow{AO}+\overrightarrow{OC}\right)\\ &= \overrightarrow{OC}\bullet\overrightarrow{OC}-\overrightarrow{AO}\bullet\overrightarrow{AO}\\ &= \left\|\overrightarrow{OC}\right\|^2-\left\|\overrightarrow{AO}\right\|^2\\ &= R^2-R^2=0\end{aligned}$$

where R is the radius of the circle and so $\overrightarrow{AC}$ is orthogonal to $\overrightarrow{BC}$, that is, the line segment AC and BC are perpendicular.
3. Find an equation of the plane passing through $A\,(6,-1,1)$, $B\,(1,0,0)$ and $C\,(21,3,2)$.
Solution:
A point of the plane is $B\,(1,0,0)$ and a normal of the plane is

$$\begin{aligned}\overrightarrow{n} &= \frac{1}{5}\overrightarrow{AB}\times\overrightarrow{AC}=\frac{1}{5}[-5,1,-1]^T\times[15,4,1]^T\\ &= \frac{1}{5}\left[\begin{vmatrix}1 & -1\\ 4 & 1\end{vmatrix},-\begin{vmatrix}-5 & -1\\ 15 & 1\end{vmatrix},\begin{vmatrix}-5 & 1\\ 15 & 4\end{vmatrix}\right]^T\\ &= \frac{1}{5}[5,-10,-35]^T=[1,-2,-7]^T\end{aligned}$$

Thus, an equation of the plane is $(x-1)-2(y-0)-7(z-0)=0$ or $x-2y-7z=1$.

4. Let A,B and C be the vertices of a triangle, and let M be the midpoint of BC. Given $A\,(2,-1,3)$, $\overrightarrow{BC}=[4,1,-2]^T$, and $\overrightarrow{AM}=[2,0,-1]^T$. Find the coordinates of B and C.
Solution:
$\overrightarrow{OB}=\overrightarrow{OA}+\overrightarrow{AM}+\overrightarrow{MB}=\overrightarrow{OA}+\overrightarrow{AM}-\frac{1}{2}\overrightarrow{BC}=[2,-1,3]^T+[2,0,-1]^T-\frac{1}{2}[4,1,-2]^T=\left[2,\frac{-3}{2},3\right]^T$ and
$\overrightarrow{OC}=\overrightarrow{OB}+\overrightarrow{BC}=\left[2,\frac{-3}{2},3\right]^T+[4,1,-2]^T=\left[6,\frac{-1}{2},1\right]^T$.

Thus, the coordinates of B and C are $B\left(2,\frac{-3}{2},3\right)$ and $C\left(6,\frac{-1}{2},1\right)$.