MATHEMATICS 221 L04/12/22 FALL 2003 QUIZ 2 SOLUTION
Thursday, October 9, 2000 at 11:00

1. Let $A = \begin{bmatrix} 3 & 5 & 1 \\ 1 & 2 & -1 \\ -1 & 0 & 1 \end{bmatrix}$.

(a) Find A^{-1} if A is invertible.

$$\left[\begin{array}{rrr|rrr} 3 & 5 & 1 & 1 & 0 & 0 \\ 1 & 2 & -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 - R_3 \\ R_2 + R_3 \\ \end{array} \left[\begin{array}{rrr|rrr} 4 & 5 & 0 & 1 & 0 & -1 \\ 0 & 2 & 0 & 0 & 1 & 1 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \tfrac{1}{2}R_2$$

$$\left[\begin{array}{rrr|rrr} 4 & 5 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] R_1 - 5R_2 \left[\begin{array}{rrr|rrr} 4 & 0 & 0 & 1 & \frac{-5}{2} & \frac{-7}{2} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \tfrac{1}{4}R_1$$

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & \frac{1}{4} & \frac{-5}{8} & \frac{-7}{8} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] R_3 + R_1 \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & \frac{1}{4} & \frac{-5}{8} & \frac{-7}{8} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{-5}{8} & \frac{1}{8} \end{array}\right]$$

Thus, A is invertible and $A^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{-5}{8} & \frac{-7}{8} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{-5}{8} & \frac{1}{8} \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 2 & -5 & -7 \\ 0 & 4 & 4 \\ 2 & -5 & 1 \end{bmatrix}$

(b) Solve the system (you may want to use the result in part (a)):

$$\begin{array}{rcrcrcr} 3x & + & 5y & + & z & = & 0 \\ x & + & 2y & - & z & = & 1 \\ -x & & & + & z & = & -1 \end{array}$$

Solution: This system is $AX = B$ where A is as in (a) and $B = [0, 1, -1]^T$, and so

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}B = \frac{1}{8} \begin{bmatrix} 2 & -5 & -7 \\ 0 & 4 & 4 \\ 2 & -5 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 2 \\ 0 \\ -6 \end{bmatrix}.$$

Thus, $x = \frac{1}{4}$, $y = 0$ and $z = \frac{-3}{4}$.

2. Find the matrix A given that $(A^{-1} - 3I)^T = 5 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Solution: Taking the transpose of both sides we have $A^{-1} - 3I = 5 \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$. Thus, $A^{-1} = 5 \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} + 3I = \begin{bmatrix} 8 & 15 \\ 10 & 23 \end{bmatrix}$ and so, $A = (A^{-1})^{-1} = \frac{1}{8 \times 23 - 15 \times 10} \begin{bmatrix} 23 & -15 \\ -10 & 8 \end{bmatrix} = \frac{1}{34} \begin{bmatrix} 23 & -15 \\ -10 & 8 \end{bmatrix}$

3. Let $A = \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}$. Express A as a product of elementary matrices.

Solution: $\begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix} \begin{array}{c} R_1 \leftrightarrow R_2 \\ E_1 \end{array} \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix} \begin{array}{c} E_2 \\ R_2 - 3R_1 \end{array} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{array}{c} E_3 \\ (-1)\,R_1 \end{array} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Thus, $E_3E_2E_1A = I$ and so $A = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

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Thursday, October 9, 2000 at 16:00

[7] **1**. Let $A = \begin{bmatrix} 2 & 1 & 2 \\ -3 & -1 & -1 \\ 5 & 2 & 1 \end{bmatrix}$.

(a) Find A^{-1} if A is invertible.

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 2 & 1 & 0 & 0 \\ -3 & -1 & -1 & 0 & 1 & 0 \\ 5 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} R_1 + R_2 \\ \\ R_3 + 2R_2 \end{array} \left[\begin{array}{ccc|ccc} -1 & 0 & 1 & 1 & 1 & 0 \\ -3 & -1 & -1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 2 & 1 \end{array}\right] \begin{array}{c} R_1 + R_3 \\ R_2 - R_3 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} -2 & 0 & 0 & 1 & 3 & 1 \\ -2 & -1 & 0 & 0 & -1 & -1 \\ -1 & 0 & -1 & 0 & 2 & 1 \end{array}\right] \begin{array}{c} \frac{-1}{2} R_1 \\ (-1)\, R_2 \\ (-1)\, R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{-1}{2} & \frac{-3}{2} & \frac{-1}{2} \\ 2 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & -2 & -1 \end{array}\right] \begin{array}{c} \\ R_2 - 2R_1 \\ R_3 - R_1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{-1}{2} & \frac{-3}{2} & \frac{-1}{2} \\ 0 & 1 & 0 & 1 & 4 & 2 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} \end{array}\right]. \text{ Thus, } A^{-1} = \frac{1}{2}\begin{bmatrix} -1 & -3 & -1 \\ 2 & 8 & 4 \\ 1 & -1 & -1 \end{bmatrix}.$$

(b) Solve the system (you may want to use the result in part (a)):

$$\begin{array}{rcrcrcl} 2x & + & y & + & 2z & = & 1 \\ -3x & - & y & - & z & = & 0 \\ 5x & + & 2y & + & z & = & 1 \end{array}$$

Solution: This system is $AX = B$ where A is as in (a) and $B = [1, 0, 1]^T$, and so

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}B = \frac{1}{2}\begin{bmatrix} -1 & -3 & -1 \\ 2 & 8 & 4 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -2 \\ 6 \\ 0 \end{bmatrix}.$$

Thus, $x = -1$, $y = 3$ and $z = 0$.

2. Find the matrix A given that $\left(2\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix} - 5A^{-1}\right)^T = \left(4A^T\right)^{-1}$.

Solution: Since $\left(4A^T\right)^{-1} = \frac{1}{4}\left(A^T\right)^{-1} = \frac{1}{4}\left(A^{-1}\right)^T = \left(\frac{1}{4}A^{-1}\right)^T$, the equation becomes: $\left(2\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix} - 5A^{-1}\right)^T = \left(\frac{1}{4}A^{-1}\right)^T$. Taking the transpose, we get $2\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix} - 5A^{-1} = \frac{1}{4}A^{-1}$. Thus, $\frac{21}{4}A^{-1} = 2\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}$, and so $A^{-1} = \frac{8}{21}\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}$. Therefore, $A = \left(A^{-1}\right)^{-1} = \frac{21}{8} \times \frac{1}{5}\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} = \frac{21}{40}\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$

3. Let $A = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}$. Express A as a product of elementary matrices.

Solution: $\begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix} \begin{array}{c} R_1 - R_2 \\ E_1 \end{array} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{array}{c} E_2 \\ R_2 - R_1 \end{array} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{array}{c} E_3 \\ (-1)\, R_1 \end{array} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Thus, $E_3E_2E_1A = I$ and so $A = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$