MATHEMATICS 221 L03/11 B09/10 FALL 2004
QUIZ 1 SOLUTION Wednesday, September 22, 2004 at 15:00

[10] **1**. Find conditions on the constant a so that the following system has (i) no solution, (ii) exactly one solution or (ii) infinitely many solutions.

$$\begin{array}{rcrcrcr} x & + & ay & - & z & = & 1 \\ -x & + & (a-2)\,y & + & z & = & -1 \\ 2x & + & 2y & + & (a-2)\,z & = & 1 \end{array}$$

Solution:

$$\begin{bmatrix} 1 & a & -1 & 1 \\ -1 & a-2 & 1 & -1 \\ 2 & 2 & a-2 & 1 \end{bmatrix} \begin{matrix} \\ R_2+R_1 \\ R_3-2R_1 \end{matrix} \begin{bmatrix} 1 & a & -1 & 1 \\ 0 & 2a-2 & 0 & 0 \\ 0 & 2-2a & a & -1 \end{bmatrix} R_3+R_2$$
$$\begin{bmatrix} 1 & a & -1 & 1 \\ 0 & 2a-2 & 0 & 0 \\ 0 & 0 & a & -1 \end{bmatrix}$$

Consider the numbers $2a-2$ and a, we have three cases:

Case 1: $2a-2=0$, that is, $a=1$. In this case, the matrix becomes

$$\begin{bmatrix} 1 & a & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} R_3 \longleftrightarrow R_2 \begin{bmatrix} 1 & a & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which says that the system has infinitely many solutions.

Case 2: $a=0$, that is, $a=0$. In this case, the matrix becomes

$$\begin{bmatrix} 1 & a & -1 & 1 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

whose last row says that the system has no solutions.

Case 3: $2a-2 \neq 0$ and $a \neq 0$, that is, $a \neq 1$ and $a \neq 0$. In this case, the matrix becomes

$$\begin{bmatrix} 1 & a & -1 & 1 \\ 0 & 2a-2 & 0 & 0 \\ 0 & 0 & a & -1 \end{bmatrix} \begin{matrix} \\ \frac{1}{2a-2}R_2 \\ \frac{1}{a}R_3R_2 \end{matrix} \begin{bmatrix} 1 & a & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{-1}{a} \end{bmatrix}$$

which says that the system has exactly one solutions.

Answer:
(i) The system has no solution when $a=0$.
(ii) The system has exactly one solution when $a \neq 1$ and $a \neq 0$.
(ii) The system has infinitely many solutions when $a=1$.

[5] **2**. Solve the systems:

$$\begin{array}{rcrcrcrcrcr} 2x & - & 2y & - & 4z & - & u & + & 7v & = & 2 \\ 4x & - & 4y & - & 7z & - & 2u & + & 16v & = & 7 \\ -x & + & y & + & 2z & - & u & - & 4v & = & -3 \end{array}$$

Solution:

$$\begin{bmatrix} 2 & -2 & -4 & -1 & 7 & 2 \\ 4 & -4 & -7 & -2 & 16 & 7 \\ -1 & 1 & 2 & -1 & -4 & -3 \end{bmatrix} \begin{matrix} R_1+R_3 \\ R_2+4R_3 \end{matrix} \begin{bmatrix} 1 & -1 & -2 & -2 & 3 & -1 \\ 0 & 0 & 1 & 2 & 0 & -5 \\ -1 & 1 & 2 & -1 & -4 & -3 \end{bmatrix} R_3+R_1$$

$$\begin{bmatrix} 1 & -1 & -2 & -2 & 3 & -1 \\ 0 & 0 & 1 & 2 & 0 & -5 \\ 0 & 0 & 0 & -3 & -1 & -4 \end{bmatrix} R_1+2R_2 \begin{bmatrix} 1 & -1 & 0 & 2 & 3 & -11 \\ 0 & 0 & 1 & 2 & 0 & -5 \\ 0 & 0 & 0 & -3 & -1 & -4 \end{bmatrix} \begin{matrix} \frac{-1}{3}R_3 \\ \\ \frac{-1}{3}R_3 \end{matrix}$$

$$\begin{bmatrix} 1 & -1 & 0 & 2 & 3 & -11 \\ 0 & 0 & 1 & 2 & 0 & -5 \\ 0 & 0 & 0 & 1 & \frac{1}{3} & \frac{4}{3} \end{bmatrix} \begin{matrix} R_1-2R_3 \\ R_2-2R_3 \end{matrix} \begin{bmatrix} 1 & -1 & 0 & 0 & \frac{7}{3} & -\frac{41}{3} \\ 0 & 0 & 1 & 0 & -\frac{2}{3} & -\frac{23}{3} \\ 0 & 0 & 0 & 1 & \frac{1}{3} & \frac{4}{3} \end{bmatrix}$$

Thus, the solutions are $\begin{bmatrix} y \\ y \\ z \\ u \\ v \end{bmatrix} = \begin{bmatrix} s-\frac{7}{3}t-\frac{41}{3} \\ s \\ \frac{2}{3}t-\frac{23}{3} \\ -\frac{1}{3}t+\frac{4}{3} \\ t \end{bmatrix}$ where s and t are any numbers.

[5] **3**. Find a matrix A so that $\left(2A - \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}\right)^T = \begin{bmatrix} -5 & 3 \\ 2 & -2 \end{bmatrix}$

Solution:

Transpose both sides of the above equation, we have

$2A - \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 3 & -2 \end{bmatrix}$ and so,

$2A = \begin{bmatrix} -5 & 2 \\ 3 & -2 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} -6 & 4 \\ 4 & -4 \end{bmatrix}$. Thus,

$A = \frac{1}{2}\begin{bmatrix} -6 & 4 \\ 4 & -4 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 2 & -2 \end{bmatrix}$.

MATHEMATICS 221 L03/11 B12/13 FALL 2004
QUIZ 1 Friday, September 24, 2004 at 11:00

1. Find conditions on the constants a and b so that the following system has (i) no solution, (ii) exactly one solution or (iii) infinitely many solutions.

$$\begin{array}{rcrcrcr} -x & + & 3y & + & 2z & = & -8 \\ x & + & & & z & = & 2 \\ 3x & + & 3y & + & az & = & b \end{array}$$

Solution:

$$\begin{bmatrix} -1 & 3 & 2 & -8 \\ 1 & 0 & 1 & 2 \\ 3 & 3 & a & b \end{bmatrix} \begin{array}{c} R_1+R_2 \\ \\ R_3-3R_2 \end{array} \begin{bmatrix} 0 & 3 & 3 & -6 \\ 1 & 0 & 1 & 2 \\ 0 & 3 & a-3 & b-6 \end{bmatrix} R_3-R_1$$

$$\begin{bmatrix} 0 & 3 & 3 & -6 \\ 1 & 0 & 1 & 2 \\ 0 & 0 & a-6 & b \end{bmatrix} \tfrac{1}{3}R_1 \begin{bmatrix} 0 & 1 & 1 & -2 \\ 1 & 0 & 1 & 2 \\ 0 & 0 & a-6 & b \end{bmatrix} R_1 \longleftrightarrow R_2$$

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & a-6 & b \end{bmatrix}$$

Consider the number $a-6$, we have two cases:

Case 1: $a-6 \neq 0$, that is, $a \neq 6$. Do $\frac{1}{a-6}R_3$, we get

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & \frac{b}{a-6} \end{bmatrix}$$ which says that the system has exactly one solution.

Case 2: $a-6=0$, that is, $a=6$. The matrix becomes: $\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & b \end{bmatrix}$. Now, we have two subcases:

Subcase 2a: $b=0$. The matrix becomes:

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ which says that the system has infinitely many solutions.

Subcase 2b: $b \neq 0$. The last row says that the system has no solutions.

Answer:

(i) The system has no solution when $a=6$ and $b \neq 0$.
(ii) The system has exactly one solution when $a \neq 6$
(ii) The system has infinitely many solutions when $a=6$ and $b=0$.

2. Solve the systems:

$$\begin{array}{rcrcrcrcrcr} x & & & - & z & + & 2u & + & w & = & 2 \\ -2x & + & y & + & 2z & - & u & & & = & -7 \\ x & + & y & - & z & + & 3u & + & w & = & -1 \end{array}$$

Solution:

$$\left[\begin{array}{rrrrrr} 1 & 0 & -1 & 2 & 1 & 2 \\ -2 & 1 & 2 & -1 & 0 & -7 \\ 1 & 1 & -1 & 3 & 1 & -1 \end{array}\right] \begin{array}{c} \\ R_2 + 2R_1 \\ R_3 - R_1 \end{array} \left[\begin{array}{rrrrrr} 1 & 0 & -1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 3 & 2 & -3 \\ 0 & 1 & 0 & 1 & 0 & -3 \end{array}\right] \begin{array}{c} \\ \\ R_3 - R_2 \end{array}$$

$$\left[\begin{array}{rrrrrr} 1 & 0 & -1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 3 & 2 & -3 \\ 0 & 0 & 0 & -2 & -2 & 0 \end{array}\right] \begin{array}{c} \\ \\ \frac{-1}{2}R_3 \end{array} \left[\begin{array}{rrrrrr} 1 & 0 & -1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 3 & 2 & -3 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right] \begin{array}{c} R_1 - 2R_3 \\ R_2 - 2R_3 \\ \\ \end{array}$$

$$\left[\begin{array}{rrrrrr} 1 & 0 & -1 & 0 & -1 & 2 \\ 0 & 1 & 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

Thus

$$\begin{array}{rcl} x & = & s + t + 2 \\ y & = & t - 3 \\ z & = & s \\ u & = & -t \\ w & = & t \end{array} \qquad \text{where } s \text{ and } t \text{ are any real numbers.}$$

3. Find a matrix A so that $\left(A - 2\begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}\right)^T = \begin{bmatrix} 2 & 3 \\ 3 & -2 \end{bmatrix}$.

Solution:

Transpose both sides of the above equation, we have

$A - 2\begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & -2 \end{bmatrix}$, and so

$$A = \begin{bmatrix} 2 & 3 \\ 3 & -2 \end{bmatrix} + 2\begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 7 \\ 5 & -6 \end{bmatrix}$$