MATHEMATICS 221 L03/11 FALL 2004
QUIZ 2 SOLUTION
Wednesday, October 6, 2004 at 15:00

1. Let $A = \begin{bmatrix} 2 & 1 & 2 \\ -3 & -1 & -1 \\ 5 & 2 & 1 \end{bmatrix}$. Find A^{-1}. Make sure that you show all the work.

Solution:

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 2 & 1 & 0 & 0 \\ -3 & -1 & -1 & 0 & 1 & 0 \\ 5 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3+2R_2]{R_1+R_2} \left[\begin{array}{cccccc} -1 & 0 & 1 & 1 & 1 & 0 \\ -3 & -1 & -1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 2 & 1 \end{array}\right] \begin{array}{c} R_1+R_3 \\ R_2-R_3 \\ \longrightarrow \end{array}$$

$$\left[\begin{array}{cccccc} -2 & 0 & 0 & 1 & 3 & 1 \\ -2 & -1 & 0 & 0 & -1 & -1 \\ -1 & 0 & -1 & 0 & 2 & 1 \end{array}\right] \begin{array}{c} -R_1 \\ -R_2 \\ -R_3 \end{array} \left[\begin{array}{cccccc} 2 & 0 & 0 & -1 & -3 & -1 \\ 2 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & -2 & -1 \end{array}\right] \begin{array}{c} \longrightarrow \\ R_2-R_1 \\ R_3-\frac{1}{2}R_1 \end{array}$$

$$\left[\begin{array}{cccccc} 2 & 0 & 0 & -1 & -3 & -1 \\ 0 & 1 & 0 & 1 & 4 & 2 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{cccccc} 1 & 0 & 0 & -\frac{1}{2} & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & -1 & 4 & 2 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{array}\right]$$

$$\text{Thus, } A^{-1} = \begin{bmatrix} \frac{-1}{2} & \frac{-3}{2} & \frac{-1}{2} \\ -1 & 4 & 2 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 & -3 & -1 \\ -2 & 8 & 4 \\ 1 & -1 & -1 \end{bmatrix}$$

2. Find the matrix A given that $\left(2A^T - 3I\right)^{-1} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$.

Solution:

$$\begin{aligned} \left(2A^T - 3I\right)^{-1} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} &\Leftrightarrow 2A^T - 3I = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} && \text{by taking inverses} \\ &\Leftrightarrow 2A^T = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} + 3I \\ &\Leftrightarrow A^T = \frac{1}{2}\left(\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} + 3I\right) \\ &\Leftrightarrow A^T = \frac{1}{2}\begin{bmatrix} 4 & -2 \\ -1 & 6 \end{bmatrix} \\ &\Leftrightarrow A = \frac{1}{2}\begin{bmatrix} 4 & -1 \\ -2 & 6 \end{bmatrix} && \text{by transposing} \end{aligned}$$

3. Let A be a square matrix. Prove or disprove each of the following statements:
(a) If $A^2 = A$ and $A \neq 0$ then A is invertible.

Solution: This statement is false because with $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, we see that $A^2 = A$ and $A \neq 0$, but A is not invertible.

(b) If $A^3 = 3I$ then A is invertible.

Solution: This statement is true. Suppose that A is a square matrix so that $A^3 = 3I$. Then $A\left(\frac{1}{3}A^3\right) = \frac{1}{3}A^3 = \frac{1}{3}(3I) = I$ and so A is invertible.

Friday, October 8, 2004 at 11:00

1. Let $A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & -1 & 1 \\ 4 & 2 & 1 \end{bmatrix}$. Find A^{-1}. Make sure that you show all the work.

Solution:

$$\left[\begin{array}{cccccc} 2 & 1 & 3 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ 4 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} R_1 + R_2 \\ \longrightarrow \\ R_3 + 2R_2 \end{array} \left[\begin{array}{cccccc} 5 & 0 & 4 & 1 & 1 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ 10 & 0 & 3 & 0 & 2 & 1 \end{array}\right] \begin{array}{c} \\ \longrightarrow \\ R_3 - R_1 \end{array}$$

$$\left[\begin{array}{cccccc} 5 & 0 & 4 & 1 & 1 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ 5 & 0 & -1 & -1 & 1 & 1 \end{array}\right] \begin{array}{c} R_1 + 4R_3 \\ R_2 + R_3 \\ \longrightarrow \end{array} \left[\begin{array}{cccccc} 25 & 0 & 0 & -3 & 5 & 4 \\ 8 & -1 & 0 & -1 & 2 & 1 \\ 5 & 0 & -1 & -1 & 1 & 1 \end{array}\right] \begin{array}{c} \frac{1}{25}R_1 \\ \longrightarrow \end{array}$$

$$\left[\begin{array}{cccccc} 1 & 0 & 0 & -\frac{3}{25} & \frac{1}{5} & \frac{4}{25} \\ 8 & -1 & 0 & -1 & 2 & 1 \\ 5 & 0 & -1 & -1 & 1 & 1 \end{array}\right] \begin{array}{c} \longrightarrow \\ R_2 - 8R_1 \\ R_3 - 5R_1 \end{array} \left[\begin{array}{cccccc} 1 & 0 & 0 & -\frac{3}{25} & \frac{1}{5} & \frac{4}{25} \\ 0 & -1 & 0 & -\frac{1}{25} & \frac{2}{5} & -\frac{7}{25} \\ 0 & 0 & -1 & -\frac{2}{5} & 0 & \frac{1}{5} \end{array}\right] \begin{array}{c} \longrightarrow \\ -R_2 \\ -R_3 \end{array}$$

$$\left[\begin{array}{cccccc} 1 & 0 & 0 & -\frac{3}{25} & \frac{1}{5} & \frac{4}{25} \\ 0 & 1 & 0 & \frac{1}{25} & -\frac{2}{5} & \frac{7}{25} \\ 0 & 0 & 1 & \frac{2}{5} & 0 & -\frac{1}{5} \end{array}\right]$$

Thus, $A^{-1} = \begin{bmatrix} -\frac{3}{25} & \frac{1}{5} & \frac{4}{25} \\ \frac{1}{25} & -\frac{2}{5} & \frac{7}{25} \\ \frac{2}{5} & 0 & -\frac{1}{5} \end{bmatrix} = \frac{1}{25}\begin{bmatrix} -3 & 5 & 4 \\ 1 & -10 & 7 \\ 10 & 0 & -5 \end{bmatrix}$

[6] **2**. Find the matrix A given that $\left(A^T - 3\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}\right)^{-1} = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$

Solution: By taking inverses we have

$$\left(A^T - 3\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}\right)^{-1} = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \iff A^T - 3\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}$$

$$\iff A^T = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$$

$$\iff A^T = \frac{1}{2}\left(\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}\right).$$

$$\iff A^T = \frac{1}{2}\begin{bmatrix} 7 & -1 \\ 11 & -3 \end{bmatrix}$$

$$\iff A = \frac{1}{2}\begin{bmatrix} 7 & -1 \\ 11 & -3 \end{bmatrix}^T = \frac{1}{2}\begin{bmatrix} 7 & 11 \\ -1 & -3 \end{bmatrix}$$

3. Let A be a square matrix. Prove or disprove each of the following statements:

(a) If $A^2 = A$ then $A = 0$ or $A = I$.

Solution: This statement is false because when $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, we have $A^2 = A$, but $A \neq 0$ and $A \neq I$.

(b) If $A^2 = 0$ then $I - A$ is invertible and $(I - A)^{-1} = I + A$.

Solution: This statement is true. Let A be a square matrix so that $A^2 = 0$. Then $(I - A)(I + A) = I^2 - A + A - A^2 = I - A^2 = I$, and so $I - A$ is invertible and $(I - A)^{-1} = I + A$.