

MATH 221 PRACTICE PROBLEMS

By W. Keith. Nicholson
Copyright 2002

1. Find A if: (a) $2A - [1 \ -3] = \left(\begin{bmatrix} 6 \\ 5 \end{bmatrix} - 3A^T \right)^T$; (b) $2A^T + \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix} = \left(\begin{bmatrix} 5 & 1 \\ -1 & 6 \end{bmatrix} - 3A \right)^T$
2. Find A in terms of B if: (a) $(2A - B)^T = A^T + (3B)^T$; (b) $(B^T - 3A)^T = 5A^T + 6B$
3. Show that every 1×3 matrix A can be written in the form

$$A = a [1 \ 0 \ 0] + b [0 \ 1 \ 0] + c [0 \ 0 \ 1]$$

for some scalars a , b and c . What can you say about 3×1 matrices?

4. If $A = -A$ where A is an $m \times n$ matrix, show that $A = 0$.
5. If A is a symmetric matrix, show that cA is also symmetric for any scalar c .
6. Show that $(-A)^T = -A^T$ for any matrix A .
7. If A and B are symmetric, show that $A - B$ is also symmetric.
8. A square matrix A is called **skew-symmetric** if $A^T = -A$.
 - (a) Show that every 2×2 skew-symmetric matrix has the form $\begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$ for some scalar b .
 - (b) If A and B are skew-symmetric, show that $A + B$ and cA are skew-symmetric for any scalar c .
9. Show that any square matrix A can be written in the form $A = S + W$ where S is symmetric and W is skew-symmetric. [*Hint*: First verify the identity $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$.]
10. In each case find the solution of the system whose augmented matrix has been carried to the following matrix R by row operations.

$$(a) R = \begin{bmatrix} 1 & 2 & 0 & 3 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(b) R = \begin{bmatrix} 1 & 0 & 7 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -3 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

11. In each case find the rank of the given matrix, possibly in terms of the parameter a .

$$(a) \begin{bmatrix} 1 & -1 & 3 & 5 \\ 3 & -2 & 1 & -2 \\ -1 & 1 & 1 & -1 \end{bmatrix}$$

$$(b) \begin{bmatrix} 1 & -4 & -5 & 2 \\ 1 & 6 & 3 & 4 \\ 1 & 1 & -1 & 3 \end{bmatrix}$$

$$(c) \begin{bmatrix} -1 & 3 & -2 & -2 \\ 3 & 1 & 1 & 9 \\ 1 & 7 & -3 & a \end{bmatrix}$$

$$(d) \begin{bmatrix} 1 & -2 & -5 & 3 \\ 2 & -3 & -8 & 7 \\ -2 & 4 & a+9 & a-7 \end{bmatrix}$$

12. If a system of 5 equations in 7 variables has a solution, explain why there is more than one solution.
13. Suppose a system of 4 equations in 4 variables has a leading 1 in each row of the row-echelon form of its augmented matrix. Must there be a unique solution? Explain.
14. The graph of a linear equation $ax + by + cz = d$ is a plane in space. By examining the possible positions of three planes in space, explain geometrically why 3 equations in 3 variables must have zero, one or infinitely many solutions.
15. If A is carried to B by a row operation, show that B can be carried back to A by another row operation, and describe the new operation in terms of the original one.

16. Find a sequence of row operations carrying
$$\begin{bmatrix} b_1 + c_1 & b_2 + c_2 & b_3 + c_3 \\ c_1 + a_1 & c_2 + a_2 & c_3 + a_3 \\ a_1 + b_1 & a_2 + b_2 & a_3 + b_3 \end{bmatrix} \rightarrow \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

17. The graph of the equation $x^2 + y^2 + ax + by + c = 0$ is a circle for any choice of the numbers a , b and c . Find the circle through the three points $(1, 2)$, $(3, -1)$ and $(0, -1)$.
18. Find the quadratic equation $f(x) = a + bx + cx^2$ which passes through the points $(0, 1)$, $(1, 2)$ and $(2, 9)$. [This is called the **interpolating polynomial** for the three data points. It is used to find data points between given ones, and in plotting curves on computer monitors.]
19. In each case find all values of a for which the system has nontrivial solutions, and determine all solutions in each case.

<p>(a) $x_1 - 2x_2 + x_3 = 0$ $x_1 + ax_2 - 3x_3 = 0$ $-x_1 + 6x_2 - 5x_3 = 0$</p>	<p>(b) $x_1 + 2x_2 + x_3 = 0$ $x_1 + 3x_2 + 6x_3 = 0$ $2x_1 + 3x_2 + ax_3 = 0$</p>
<p>(c) $x_1 + x_2 - x_3 = 0$ $ax_2 - 2x_3 = 0$ $x_1 + x_2 + ax_3 = 0$</p>	<p>(d) $ax_1 + x_2 + x_3 = 0$ $x_1 + x_2 - x_3 = 0$ $x_1 + x_2 + ax_3 = 0$</p>

20. Consider the matrices $A = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$, and $C = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$.

(a) Show that the only choice of numbers x , y and z such that $xA + yB + zC = 0$ is $x = y = z = 0$. Because of this we say that the set $\{A, B, C\}$ of matrices is **linearly independent**. We will have more to say about this important notion in Chapter 4.

(b) Is the set $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ linearly independent? Support your answer.

21. Show algebraically that there is a line through any two points in the plane. [*Hint*: Use the fact that every line has equation $ax + by + c = 0$ where a , b and c are not all zero.]

22. Every plane in space has equation $ax + by + cz + d = 0$ where a , b and c are not all zero. Show algebraically that there is a plane through any three points in space. [*Hint*: Preceding exercise.]

23. Find all solutions to the following system:

$$\begin{aligned}x + y + 2z &= 2 \\2x + y - z &= 3 \\x + 2y + 7z &= 3\end{aligned}$$

24. Find the augmented matrix, in reduced row-echelon form, of a system of equations in the variables x , y and z which has the following solutions: $x = 1 - 2t$, $y = -3 + t$ and $z = t$.

25. Find all solutions to the following system:

$$\begin{aligned}x + 2y + z &= -1 \\3x + 5y + z &= 2 \\-x - y + 3z &= -5\end{aligned}$$

26. Find all solutions to the following system:

$$\begin{aligned}x_1 - x_2 + 2x_4 + x_5 &= 2 \\-2x_1 + 2x_2 + x_3 - 4x_4 &= -7 \\x_1 - x_2 + x_3 + 3x_4 + x_5 &= -1\end{aligned}$$

27. Find (if possible) conditions on the numbers a , b and c so that the following set of linear equations has no solution, a unique solution, or infinitely many solutions.

$$\begin{aligned}x - y + 2z &= a \\2x - y + 3z &= b \\-x + 2y - 3z &= c\end{aligned}$$

28. Find conditions on a such that the system

$$\begin{aligned}x - y + 2z &= a \\2x + y - z &= 3 \\x + 5y - 8z &= 1\end{aligned}$$

has zero, one or infinitely many solutions.

29. Either prove the following statement or give an example showing that it is false: *If there is more than one solution to a system of linear equations, the augmented matrix A of the system has a row of zeros.*

30. Find all solutions to the system:

$$\begin{aligned}x_1 - x_2 + 2x_3 + 2x_4 + 3x_5 &= -4 \\-2x_1 + 3x_2 - 6x_3 - 3x_4 - 11x_5 &= 11 \\-x_1 + 2x_2 - 4x_3 + x_4 - 8x_5 &= 7 \\x_2 - 2x_3 + 3x_4 - 5x_5 &= 3\end{aligned}$$

31. Find the augmented matrix, in reduced row-echelon form, of a system of three equations in five variables x_1 , x_2 , x_3 , x_4 , and x_5 , with solutions $x_1 = 2t - s - 2$, $x_2 = 3$, $x_3 = s$, $x_4 = 6 - t$, and $x_5 = t$.

32. Simplify the following expressions where A , B and C represent matrices.
- $A(3B - C) + (A - 2B)C + 2B(C + 2A)$
 - $A(B + C - D) + B(C - A + D) - (A + B)C + (A - B)D$
 - $AB(BC - CB) + (CA - AB)BC + CA(A - B)C$
 - $(A - B)(C - A) + (C - B)(A - C) + (C - A)^2$
33. If A is a real symmetric 2×2 matrix and $A^2 = 0$, show that $A = 0$. Give an example to show that it is essential that A is symmetric.
34. If $A = \begin{bmatrix} a & b & c \\ a_1 & b_1 & c_1 \end{bmatrix}$ and $AA^T = 0$, show that $A = 0$. [Remark: More generally, if A is any matrix such that $AA^T = 0$, then necessarily $A = 0$.]
35. If A is any matrix, show that AA^T is a symmetric matrix.
36. If A and B are matrices that both commute with a matrix C , show that the matrix $2A - 3B$ also commutes with C .
37. Find the matrix A if $\left[A^T - 3 \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \right]^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.
38. Find the matrix A if $[A - 2I]^{-1} = A^{-1} \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix}$.
39. If A is a square matrix and $AX = 0$ for some matrix $X \neq 0$, show that A has no inverse.
40. If $U = \begin{bmatrix} 3 & -4 \\ 7 & 5 \end{bmatrix}$ and $AU = 0$ for some matrix A , show that necessarily $A = 0$.
41. If A and B are $n \times n$ matrices such that AB and B are both invertible, show that A is also invertible using *only* Theorem 3 §1.5.
42. If A and B are $n \times n$ matrices and $AB = cI$ where $c \neq 0$, show that $BA = cI$. Is it true if $c = 0$?
43. Let A be a square matrix which satisfies $A^3 - 2A^2 + 5A + 6I = 0$. Show that A is invertible, and find a formula for A^{-1} in terms of A .
44. If $E^2 = E$ and $A = I - 2E$, show that $A^{-1} = A$.
45. Find the inverse of $\begin{bmatrix} 1 & -1 & -1 \\ 2 & -1 & -4 \\ 1 & -2 & 2 \end{bmatrix}$.
46. If the first row of a square matrix A consists of zeros, show that A does not have an inverse.
47. If A is an invertible $n \times n$ matrix, show that $AX = B$ has a unique solution for any $n \times k$ matrix B .

48. If $\det \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix} = 5$, find $\det \begin{bmatrix} a + 2x & b + 2y & c + 2z \\ x + p & y + q & z + r \\ 3p & 3q & 3r \end{bmatrix}$.

49. Find the values of the number c such that $\begin{bmatrix} 1 & c & 0 \\ 2 & 0 & c \\ c & -1 & 1 \end{bmatrix}$ has an inverse.

50. Find the inverse of $\begin{bmatrix} 1 & -1 & -2 \\ -1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$, and use it to solve $\begin{cases} x - y - 2z = 3 \\ -x + z = 0 \\ 2x + y = 1 \end{cases}$

51. Assume that $\det(A) = 3$ where $A = \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}$. Compute $\det(-2B^{-1})$ where

$$B = \begin{bmatrix} 2x & a + 2p & p - 3x \\ 2y & b + 2q & q - 3y \\ 2z & c + 2r & r - 3z \end{bmatrix}.$$

52. Show that there is no real 3×3 matrix A such that $A^2 = -I$.

53. Show that $\det \begin{bmatrix} p + x & q + y & r + z \\ a + x & b + y & c + z \\ a + p & b + q & c + r \end{bmatrix} = 2 \det \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}$.

54. Show that $\det \begin{bmatrix} 1 & a & p & q \\ x & 1 & b & r \\ x^2 & x & 1 & c \\ x^3 & x^2 & x & 1 \end{bmatrix} = (1 - ax)(1 - bx)(1 - cx)$ for any choice of p, q and r . [*Hint:*

Begin by eliminating x from column 1.]

55. In each case evaluate $\det A$ by inspection.

(a) $A = \begin{bmatrix} a & 3 - a & a + 1 \\ b & 3 - b & b + 1 \\ c & 3 - c & c + 1 \end{bmatrix}$ (b) $A = \begin{bmatrix} a & b & c \\ a + b & 2b & c + b \\ 3 & 3 & 3 \end{bmatrix}$

56. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} a + c & 2c \\ b + d & 2d \end{bmatrix}$. If $\det A = 2$, find $\det(A^2 B^T A^{-1})$.

57. Evaluate $\det \begin{bmatrix} x - 1 & 2 & 3 \\ 2 & -3 & x - 2 \\ -2 & x & -2 \end{bmatrix}$ by first adding all other rows to the first row. Then find all values of x such that the determinant is zero.

58. If A is a 4×4 matrix and $A^2 = 3A$, what are the possible values of $\det(A)$?

59. If $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = -3$, compute $\det \begin{bmatrix} 3 & -3 & 0 \\ c+5 & -5 & 3a \\ d-2 & 2 & 3b \end{bmatrix}$.
60. If A and B are $n \times n$ where n is odd, and if $AB = -BA$, show that either A or B has no inverse.
61. If A is 4×4 and $\det A = 2$, find $\det(15A^{-1} - 6 \operatorname{adj} A)$.
62. In each case: (1) Find the values of the number c such that A has an inverse, and (2) Find A^{-1} for those values of c .
- (a) $A = \begin{bmatrix} c & c & 1 \\ 1 & c & 1 \\ c & -1 & 2 \end{bmatrix}$ (b) $A = \begin{bmatrix} 4 & c & 3 \\ c & 2 & c \\ 5 & c & 4 \end{bmatrix}$
63. If $\det A = 3$, $\det B = -1$ and $\det C = 2$, compute the determinant of:
- (a) $\begin{bmatrix} A & X & Y \\ 0 & B & Z \\ 0 & 0 & C \end{bmatrix}$ (b) $\begin{bmatrix} A & X & 0 \\ 0 & B & 0 \\ Y & Z & C \end{bmatrix}$
64. If A is 2×2 and B is 3×3 , show that $\det \begin{bmatrix} 0 & B \\ A & X \end{bmatrix} = \det A \det B$. [Hint: First left multiply by $\begin{bmatrix} 0 & I_2 \\ I_3 & 0 \end{bmatrix}$.]
65. Consider the matrix $A = \begin{bmatrix} 0 & 2 \\ 2 & -3 \end{bmatrix}$. Find the characteristic polynomial, eigenvalues and eigenvectors for A , and find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.
66. Consider the matrix $A = \begin{bmatrix} -2 & 1 \\ 4 & 1 \end{bmatrix}$. Find the characteristic polynomial, eigenvalues and eigenvectors for A , and find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.
67. Show that $A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$ is not diagonalizable.
68. If $A^k = 0$ for some $k \geq 1$, show that 0 is the only eigenvalue of A .
69. If A is a diagonalizable $n \times n$ matrix and every eigenvalue of A is zero, show that $A = 0$.
70. If $A^2 = A$, show that 0 and 1 are the only eigenvalues of A .
71. If A is a diagonalizable matrix, and if every eigenvalue λ of A satisfies $\lambda^2 = \lambda$, show that $A^2 = A$.
72. If A is a diagonalizable $n \times n$ matrix, show that A^2 is also diagonalizable.
73. If A is a diagonalizable $n \times n$ matrix, show that A^T is also diagonalizable.

74. Determine whether $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 2 \end{bmatrix}$ is diagonalizable.
75. Show that $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix}$ is not diagonalizable.
76. If A is diagonalizable and $\lambda_i \geq 0$ for each eigenvalue λ_i , show that $A = B^2$ for some matrix B . [*Hint*: If $P^{-1}AP = D = \text{diag}(\lambda_1, \dots, \lambda_n)$, take $B = PD_0P^{-1}$ where $D_0 = \text{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n})$.]
77. If A is diagonalizable and has only one eigenvalue λ , show that $A = \lambda I$.
78. If A is diagonalizable with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ (possibly not all distinct), show that $\det A = \lambda_1 \lambda_2 \cdots \lambda_n$. [*Remark*: This holds for any square matrix, diagonalizable or not.]
79. Find A^{-1} if $A = \begin{bmatrix} 1 & i \\ -i & 1+i \end{bmatrix}$.
80. Find a quadratic equation with real coefficients that has $2 - 3i$ as a root. What is the other root?
81. Show that $w = 3 - 2i$ is a root of $x^2 - 6x + 13$. What is the other root? Justify your answer.
82. Show that $z = (1+i)^n + (1-i)^n$ is a real number for each $n \geq 1$ by first finding the conjugate \bar{z} .
83. If $z \neq 0$ is a complex number, show that $1/z = \frac{1}{|z|^2} \bar{z}$.
84. If zw is real and $z \neq 0$, show that $w = r \bar{z}$ for some real number r .
85. Show that $|z+w|^2 + |z-w|^2 = 2(|z|^2 + |w|^2)$ for all complex numbers z and w . [*Hint*: $|z|^2 = z\bar{z}$.]
86. Find the point $\frac{1}{5}$ the way from $P(2, -1, 5)$ to $Q(3, 0, 4)$.
87. Find the two trisection points between $P(1, 2, 3)$ and $Q(8, -2, 0)$.
88. Let A, B and C denote the vertices of a triangle. If E is the midpoint of side BC , show that $\overrightarrow{AE} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC})$. [*Hint*: Start by writing $\overrightarrow{AE} = \overrightarrow{AB} + \overrightarrow{BE}$.]
89. The unit cube has three of its vertices $O(0, 0, 0)$, $A(1, 0, 0)$, $B(0, 1, 0)$ and $C(0, 0, 1)$. Show that, of the four diagonals of the unit cube, no two are perpendicular.
90. In each case write the vector \vec{v} as a sum $\vec{v} = \vec{v}_1 + \vec{v}_2$ where \vec{v}_1 is parallel to \vec{d} and \vec{v}_2 is orthogonal to \vec{d} . (a) $\vec{v} = [3 \ -1 \ 2]^T$ and $\vec{d} = [1 \ 2 \ 1]^T$. (b) $\vec{v} = [5 \ 1 \ -2]^T$ and $\vec{d} = [3 \ 0 \ -7]^T$.
91. If $\|\vec{v}\|^2 + \|\vec{w}\|^2 = \|\vec{v} + \vec{w}\|^2$ where $\vec{v} \neq \vec{0}$ and $\vec{w} \neq \vec{0}$, show that \vec{v} and \vec{w} are orthogonal.

92. Find the scalar equations of the line through the point $P(3, -1, 2)$ which is parallel to the line $[x \ y \ z]^T = [2 - 5t \ 3 \ 2t]^T$ where t is arbitrary.
93. Find the scalar equations of the line through the points $P_1(1, 0, -2)$ and $P_2(2, 1, -1)$.
94. Find the point of intersection of the line $[x \ y \ z]^T = [2 \ -1 \ 3]^T + t[1 \ -1 \ -4]^T$ and the plane $3x + y - 2z = 4$.
95. Find the equation of the plane through the point $P(1, 1, -2)$ which contains the line $[x \ y \ z]^T = [3 \ -1 \ 0]^T + t[1 \ 1 \ -1]^T$.
96. Determine the equation of the line through the point $P(1, -1, 0)$ which is perpendicular to the plane $x + y - 2z = 3$.
97. Find the equation of the plane through the point $P_0(2, 3, -1)$ which is parallel to the plane with equation $4x - 3y + z = 4$.
98. Find the point Q on the line with equation $[x \ y \ z]^T = [1 \ 2 \ 0]^T + t[2 \ -1 \ 1]^T$ which is closest to the point $P(0, 1, 2)$.
99. Find the shortest distance from the point $P(1, 0, 2)$ to the plane $5x - 7y + 2z = 3$.
100. Find the shortest distance from the point $P(1, 0, 2)$ to the line $[x \ y \ z]^T = [1 \ -1 \ 0]^T + t[2 \ 1 \ 1]^T$.
101. Consider the the plane through the point $P_0(1, -1, 0)$ which is parallel to the plane with equation $2x - 3y + 2z = 4$. Does this plane pass through the origin? Support your answer.
102. Consider the points $A(2, 2, 1)$, $B(1, 1, 0)$ and $C(2, 3, -3)$.
- Are these points the vertices of a right-angled triangle? Justify your answer.
 - Find the cosine of the interior angle of the triangle at vertex C .
103. Find the area of the triangle with vertices $A(1, 0, 0)$, $B(0, 1, 0)$ and $C(0, 0, 1)$.
104. Consider the transformation T defined as follows:
 Rotation through $\pi/2$ followed by reflection in the line $y = x$.
 Determine the effect of T , that is determine if it is a rotation (and find the angle) or a reflection or projection in some line through the origin (and find the line).
105. Find the reflection of the point $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ in the line $y = -3x$.
106. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation with $T \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$ and $T \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.
- Find the matrix of T and give a formula for $T \begin{bmatrix} x \\ y \end{bmatrix}$.
 - Compute $T^{-1} \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.