MATHEMATICS 221 L03/11 B09/10/41/42 FALL 2004
QUIZ 5 SOLUTION
Wednesday, December 1, 2004 at 15:00

1. Let P_1 be the plane with equation $x+2y-z=2$ and P_2 be the plane with equation $2x-y+z=2$.

(a) Do the planes P_1 and P_2 intersect each other? Explain. If the planes P_1 and P_2 intersect each other, find a direction vetor of the line of intersection (of the planes P_1 and P_2).

Solution: The planes P_1 and P_2 do intersect each other. This the because the normals $\overrightarrow{n_1}=[1,2,-1]^T$ and $\overrightarrow{n_2}=[2,-1,1]^T$ of the planes are not parallel. Since the line of intersection of the planes P_1 and P_2 is perpendicular to both of the planes, a direction vector of this line can be chosen as

$$\begin{aligned}\overrightarrow{d} &= \overrightarrow{n_1}\times\overrightarrow{n_2}\\ &= [1,2,-1]^T\times[2,-1,1]^T\\ &= \left[\begin{vmatrix}2 & -1\\ -1 & 1\end{vmatrix}, -\begin{vmatrix}1 & -1\\ 2 & 1\end{vmatrix}, \begin{vmatrix}1 & 2\\ 2 & -1\end{vmatrix}\right]^T\\ &= [1,-3,-5]^T\end{aligned}$$

(b) Find an equation of the line that passes through the point $M(1,2,2)$ and is parallel to both of the planes P_1 and P_2.

Solution: A line that is parallel to both of the planes P_1 and P_2 has $\overrightarrow{d}=[1,-3,-5]^T$ as a direction vector, as seen in part (a). Thus, an equation of the line that passes through the point $M(1,2,2)$ and is parallel to both of the planes P_1 and P_2 is

$$[x,y,z]^T=[1,2,2]^T+t[1,-3,-5]^T.$$

2. Find the shortest distance between the two nonparallel lines $[x,y,z]^T=[3,1,-2]^T+t[2,1,-3]^T$ and $[x,y,z]^T=[2,3,-1]^T+t[1,1,1]^T$.

Solution: The direction of the two lines are $\overrightarrow{d_1}=[2,1,-3]^T$ and $\overrightarrow{d_2}=[1,1,1]^T$. Let

$$\begin{aligned}\overrightarrow{n} &= \overrightarrow{d_1}\times\overrightarrow{d_2}\\ &= [2,1,-3]^T\times[1,1,1]^T\\ &= \left[\begin{vmatrix}1 & -3\\ 1 & 1\end{vmatrix}, -\begin{vmatrix}2 & -3\\ 1 & 1\end{vmatrix}, \begin{vmatrix}2 & 1\\ 1 & 1\end{vmatrix}\right]^T\\ &= [4,-5,1]^T.\end{aligned}$$

Then $\overrightarrow{n}$ is a vector that is perpendicular two both lines. The points $A(3,1,-2)$ and $B(2,3,-1)$ are on these two lines and the distance between these lines is

$$\begin{aligned}\left\|proj_{\overrightarrow{n}}\overrightarrow{AB}\right\| &= \left\|\frac{\overrightarrow{AB}\bullet\overrightarrow{n}}{\|\overrightarrow{n}\|^2}\overrightarrow{n}\right\|\\ &= \frac{\left|\overrightarrow{AB}\bullet\overrightarrow{n}\right|}{\|\overrightarrow{n}\|^2}\|\overrightarrow{n}\|\\ &= \frac{\left|\overrightarrow{AB}\bullet\overrightarrow{n}\right|}{\|\overrightarrow{n}\|}\\ &= \frac{\left|[-1,2,1]^T\bullet[4,-5,1]^T\right|}{\left\|[4,-5,1]^T\right\|}\\ &= \frac{|-4-10+1|}{\sqrt{4^2+(-5)^2+1^2}}\\ &= \frac{13}{\sqrt{42}}=\frac{13}{42}\sqrt{42}\end{aligned}$$

3. Find the shortest distance from the point $P\,(3,-1,1)$ to the plane $3x-y+4z=5$, and find the point Q on the plane closest to P.
Solution: Choose a point $A\,(0,-5,0)$ on the plane $3x-y+4z=5$. Let Q be the point on the plane $3x-y+4z=5$ that is closest to P. Note that $\overrightarrow{n}=[3,-1,4]^T$ is a normal of this plane.Then

$$\begin{aligned}\overrightarrow{QP} &= proj_{\overrightarrow{n}}\overrightarrow{AP} \\ &= \frac{\overrightarrow{AP}\bullet\overrightarrow{n}}{\|\overrightarrow{n}\|^2}\overrightarrow{n} \\ &= \frac{[3,4,1]^T\bullet[3,-1,4]^T}{\left\|[3,-1,4]^T\right\|^2}[3,-1,4]^T \\ &= \frac{9}{26}[3,-1,4]^T\end{aligned}$$

The distance is $\left\|\overrightarrow{QP}\right\| = \frac{9}{26}\left\|[3,-1,4]^T\right\| = \frac{9\sqrt{26}}{26}$

Now, $\overrightarrow{OQ}=\overrightarrow{OP}+\overrightarrow{PQ}=\overrightarrow{OP}-\overrightarrow{QP}=[3,-1,1]^T-\frac{9}{26}[3,-1,4]^T=\left[\frac{51}{26},-\frac{17}{26},-\frac{10}{26}\right]^T$.
Thus, the coordinates of Q is $Q\left(\frac{51}{26},-\frac{17}{26},-\frac{10}{26}\right)$.

MATHEMATICS 221 L03/11 B11/12/43/44 FALL 2004
QUIZ 5
Friday, December 3, 2004 at 11:00

1. Let P_1 be the plane with equation $2x+y-z=2$ and P_2 be the plane with equation $2x-y+z=2$.
(a) Do the planes P_1 and P_2 intersect each other? Explain. If the planes P_1 and P_2 intersect each other, find a direction vetor of the line of intersection (of the planes P_1 and P_2).
Solution: The planes P_1 and P_2 do intersect each other. This the because the normals $\overrightarrow{n_1}=[2,1,-1]^T$ and $\overrightarrow{n_2}=[2,-1,1]^T$ of the planes are not parallel. Since the line of intersection of the planes P_1 and P_2 is perpendicular to both of the planes, a direction vector of this line can be chosen as

$$\begin{aligned}\overrightarrow{d} &= \tfrac{1}{2}\overrightarrow{n_1}\times\overrightarrow{n_2}\\ &= \tfrac{1}{4}[2,1,-1]^T\times[2,-1,1]^T\\ &= \tfrac{1}{4}\left[\begin{vmatrix}1 & -1\\ -1 & 1\end{vmatrix}, -\begin{vmatrix}2 & -1\\ 2 & 1\end{vmatrix}, \begin{vmatrix}2 & 1\\ 2 & -1\end{vmatrix}\right]^T\\ &= \tfrac{1}{4}[0,-4,-4]^T=[0,1,1]^T\end{aligned}$$

(b) Find an equation of the line that passes through the point $M(1,2,3)$ and is parallel to both of the planes P_1 and P_2.
Solution: A line that is parallel to both of the planes P_1 and P_2 has $\overrightarrow{d}=[1,-3,-5]^T$ as a direction vector, as seen in part (a). Thus, an equation of the line that passes through the point $M(1,2,2)$ and is parallel to both of the planes P_1 and P_2 is

$$[x,y,z]^T=[1,2,3]^T+t[0,1,1]^T.$$

2. Find the shortest distance between the two nonparallel lines $[x,y,z]^T=[7,8,1]^T+t[3,-2,5]^T$ and $[x,y,z]^T=[6,1,5]^T+t[1,3,2]^T$.
Solution: The direction of the two lines are $\overrightarrow{d_1}=[3,-2,5]^T$ and $\overrightarrow{d_2}=[1,3,2]^T$. Let

$$\begin{aligned}\overrightarrow{n} &= \overrightarrow{d_1}\times\overrightarrow{d_2}\\ &= [3,-2,5]^T\times[1,3,2]^T\\ &= \left[\begin{vmatrix}-2 & 5\\ 3 & 2\end{vmatrix}, -\begin{vmatrix}3 & 5\\ 1 & 2\end{vmatrix}, \begin{vmatrix}3 & -2\\ 1 & 3\end{vmatrix}\right]^T\\ &= [-19,-1,11]^T.\end{aligned}$$

Then $\overrightarrow{n}$ is a vector that is perpendicular two both lines. The points $A(7,8,1)$ and $B(6,1,5)$ are on these two lines and the distance between these lines is

$$\begin{aligned}\left\|proj_{\overrightarrow{n}}\overrightarrow{AB}\right\| &= \left\|\frac{\overrightarrow{AB}\bullet\overrightarrow{n}}{\|\overrightarrow{n}\|^2}\overrightarrow{n}\right\|\\ &= \frac{\left|\overrightarrow{AB}\bullet\overrightarrow{n}\right|}{\|\overrightarrow{n}\|^2}\|\overrightarrow{n}\|\\ &= \frac{\left|\overrightarrow{AB}\bullet\overrightarrow{n}\right|}{\|\overrightarrow{n}\|}\\ &= \frac{\left|[-1,-7,4]^T\bullet[-19,-1,11]^T\right|}{\left\|[-19,-1,11]^T\right\|}\\ &= \frac{|19+7+44|}{\sqrt{(-19)^2+(-1)^2+11^2}}\\ &= \frac{70}{\sqrt{483}}\end{aligned}$$

[6] **3**. Find the shortest distance from the point $P(2,0,3)$ to the plane $x-2y+2z=-1$, and find the point Q on the plane closest to P.

Solution: Choose a point $A(-1,0,0)$ on the plane $x-2y+2z=-1$. Let Q be the point on the plane $x-2y+2z=-1$ that is closest to P. Note that $\overrightarrow{n}=[1,-2,2]^T$ is a normal of this plane.Then

$$\begin{aligned}
\overrightarrow{QP} &= proj_{\overrightarrow{n}}\overrightarrow{AP} \\
&= \frac{\overrightarrow{AP}\bullet\overrightarrow{n}}{\|\overrightarrow{n}\|^2}\overrightarrow{n} \\
&= \frac{[3,0,3]^T\bullet[1,-2,2]^T}{\left\|[1,-2,2]^T\right\|^2}[1,-2,2]^{TT} \\
&= \frac{9}{9}[1,-2,2]^T \\
&= [1,-2,2]^T
\end{aligned}$$

The distance is $\left\|\overrightarrow{QP}\right\|=\left\|[1,-2,2]^T\right\|=3$

Now, $\overrightarrow{OQ}=\overrightarrow{OP}+\overrightarrow{PQ}=\overrightarrow{OP}-\overrightarrow{QP}=[2,0,3]^T-[1,-2,2]^T=[1,2,1]^T$. Thus, the coordinates of Q is $Q(1,2,1)$.