

## Practice Problems S2

1. Let  $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & 4 \end{bmatrix}$ ,  $B = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 9 & 7 \\ -1 & 0 & 2 \end{bmatrix}$  and  $C = \begin{bmatrix} 1 & 3 & 2 \\ 1 & 1 & 1 \\ -1 & 4 & 1 \end{bmatrix}$ .

Compute the products  $AB$ ,  $BA^T$ ,  $BC$  and  $CB$ .

2. Consider the following system of linear equations:

$$\begin{cases} x_1 - 2x_2 + x_3 - 4x_4 = 1 \\ x_1 + 3x_2 + 7x_3 + 2x_4 = 2 \\ x_1 - 12x_2 - 11x_3 - 16x_4 = -1 \end{cases}.$$

(a) Find basic solutions to the associated homogeneous system;

(b) Find a particular solution to the system.

3. Find the general solution to the linear system  $AX = B$  and specify a particular solution, where

$$A = \begin{bmatrix} 2 & 1 & -1 & -1 \\ 3 & 1 & 1 & -2 \\ -1 & -1 & 2 & 1 \\ -2 & -1 & 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 \\ -2 \\ 2 \\ 3 \end{bmatrix}.$$

Find basic solutions and write the general solution to the associated homogeneous system as a linear combination of these basic solutions.

4. Find the inverses of the following matrices:

(a)  $\begin{bmatrix} 7 & 4 \\ 3 & 2 \end{bmatrix}$ ; (b)  $\begin{bmatrix} 1 & 3 & 2 \\ 1 & 1 & 1 \\ -1 & 4 & 1 \end{bmatrix}$ .

5. Use matrix inversion to solve the following systems of linear equations:

(a)  $\begin{cases} 7x + 4y = 2 \\ 3x + 2y = -2 \end{cases}$ ; (b)  $\begin{cases} x + 3y + 2z = 5 \\ x + y + z = 1 \\ -x + 4y + z = 5 \end{cases}$ .

6. Consider a directed graph with three vertices  $v_1$ ,  $v_2$  and  $v_3$ . Find the adjacency matrix of this graph if the edges are  $v_1 \rightarrow v_1$ ,  $v_1 \rightarrow v_2$ ,  $v_2 \rightarrow v_3$ ,  $v_3 \rightarrow v_2$  and  $v_3 \rightarrow v_1$ . Determine the number of paths of length 5 from  $v_2$  to  $v_3$  and from  $v_3$  to  $v_1$ .

### Recommended Problems:

Pages 34 - 35: 1a,c; 2a,c,d,f,g; 3a,c; 4a; 5a; 8a

Pages 47 - 50: 1a,b,c,d,f,g; 2a,b; 5a; 7a,b; 8; 10; 16a; 22; 32

Pages 59 - 60: 1a,b,c; 2a,b,c,d,e,f,k; 3a,c,e; 4a; 5a,b; 6a,b.

### Solutions

$$1. AB = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 1 & 9 & 7 \\ -1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -6 & -2 \\ 0 & 6 & 10 \end{bmatrix},$$

$$BA^T = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 9 & 7 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 8 \\ 6 & 30 \\ 3 & 6 \end{bmatrix},$$

$$BC = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 9 & 7 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 1 & 1 & 1 \\ -1 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 13 & 8 \\ 3 & 40 & 18 \\ -3 & 5 & 0 \end{bmatrix},$$

$$CB = \begin{bmatrix} 1 & 3 & 2 \\ 1 & 1 & 1 \\ -1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 1 & 9 & 7 \\ -1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 30 & 26 \\ 2 & 12 & 10 \\ 1 & 33 & 29 \end{bmatrix}.$$

2. The system has augmented matrix  $\begin{bmatrix} 1 & -2 & 1 & -4 & 1 \\ 1 & 3 & 7 & 2 & 2 \\ 1 & -12 & -11 & -16 & -1 \end{bmatrix}$  which can be brought

by row operations to the reduced row-echelon matrix  $\begin{bmatrix} 1 & 0 & 17/5 & -8/5 & 7/5 \\ 0 & 1 & 6/5 & 6/5 & 1/5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ .

The system has general solution

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7/5 - 17/5t + 8/5s \\ 1/5 - 6/5t - 6/5s \\ t \\ s \end{bmatrix} = \begin{bmatrix} 7/5 \\ 1/5 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -17/5 \\ -6/5 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 8/5 \\ -6/5 \\ 0 \\ 1 \end{bmatrix}.$$

$X_1 = \begin{bmatrix} 7/5 \\ 1/5 \\ 0 \\ 0 \end{bmatrix}$  is a particular solution to the system. The associated homogeneous

system has general solution  $X_0 = t \begin{bmatrix} -17/5 \\ -6/5 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 8/5 \\ -6/5 \\ 0 \\ 1 \end{bmatrix}$ , with  $X_2 = \begin{bmatrix} -17/5 \\ -6/5 \\ 1 \\ 0 \end{bmatrix}$

and  $X_3 = \begin{bmatrix} 8/5 \\ -6/5 \\ 0 \\ 1 \end{bmatrix}$ , or  $X_4 = \begin{bmatrix} -17 \\ -6 \\ 5 \\ 0 \end{bmatrix}$  and  $X_5 = \begin{bmatrix} 8 \\ -6 \\ 0 \\ 5 \end{bmatrix}$  as basic solutions.

3. The augmented matrix of the system has reduced row-echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & -4 & -9 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, the general solution is

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 - t \\ -9 + 4t \\ -2 + t \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ -9 \\ -2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 4 \\ 1 \\ 1 \end{bmatrix}.$$

$X_1 = \begin{bmatrix} 3 \\ -9 \\ -2 \\ 0 \end{bmatrix}$  is a particular solution. The associated homogeneous system has

general solution  $X_0 = t \begin{bmatrix} -1 \\ 4 \\ 1 \\ 1 \end{bmatrix}$  with  $\begin{bmatrix} -1 \\ 4 \\ 1 \\ 1 \end{bmatrix}$  as a basic solution.

4. (a)  $\det(A) = \det \begin{vmatrix} 7 & 4 \\ 3 & 2 \end{vmatrix} = 7 \times 2 - 3 \times 4 = 2 \neq 0$ . So,  $A$  is invertible. Its inverse

$A^{-1} = \frac{1}{\det(A)} \text{adj}(A) = \frac{1}{2} \begin{bmatrix} 2 & -4 \\ -3 & 7 \end{bmatrix}$ . It can be also obtained by bringing  $[A|I_2]$  to the reduced row-echelon form  $[I_2|A^{-1}]$ .

(b) By row operations carry the matrix  $\begin{bmatrix} 1 & 3 & 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ -1 & 4 & 1 & 0 & 0 & 1 \end{bmatrix}$  to its reduced row-echelon form ( $[A|I_3] \longrightarrow [I_3|A^{-1}]$ ):

$$\begin{bmatrix} 1 & 0 & 0 & -3 & 5 & 1 \\ 0 & 1 & 0 & -2 & 3 & 1 \\ 0 & 0 & 1 & 5 & -7 & -2 \end{bmatrix}.$$

$$\text{Therefore, } A^{-1} = \begin{bmatrix} -3 & 5 & 1 \\ -2 & 3 & 1 \\ 5 & -7 & -2 \end{bmatrix}.$$

5. In matrix form:

(a)

$$\begin{aligned} \begin{bmatrix} 7 & 4 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 2 \\ -2 \end{bmatrix} \implies \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ 3 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ -2 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 2 & -4 \\ -3 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 6 \\ -10 \end{bmatrix}, \end{aligned}$$

i.e.,  $x = 6$  and  $y = -10$ .

(b)

$$\begin{aligned} \begin{bmatrix} 1 & 3 & 2 \\ 1 & 1 & 1 \\ -1 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 5 \\ 1 \\ 5 \end{bmatrix} \implies \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 3 & 2 \\ 1 & 1 & 1 \\ -1 & 4 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 1 \\ 5 \end{bmatrix} \\ &= \begin{bmatrix} -3 & 5 & 1 \\ -2 & 3 & 1 \\ 5 & -7 & -2 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} -5 \\ -2 \\ 8 \end{bmatrix}, \end{aligned}$$

i.e.,  $x = -5$ ,  $y = -2$  and  $z = 8$ .

6. This directed graph has adjacency matrix  $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ . The number of 5-paths (i.e., paths of length 5) from a vertex  $v_j$  to another vertex  $v_i$  is the  $(i, j)$ -entry of  $A^5 = \begin{bmatrix} 5 & 3 & 5 \\ 5 & 3 & 5 \\ 3 & 2 & 3 \end{bmatrix}$ . Therefore, there are 2 5-paths from  $v_2$  to  $v_3$  and 5 5-paths from  $v_3$  to  $v_1$ .