

## Practice Problems S5 (Diagonalization, Linear Recurrences)

1. Compute  $P^{-1}AP$  and then  $A^n$  if  $A = \begin{bmatrix} 6 & -5 \\ 2 & -1 \end{bmatrix}$  and  $P = \begin{bmatrix} 5 & 1 \\ 2 & 1 \end{bmatrix}$
2. (Diagonalization) Find the characteristic polynomial, eigenvalues and an invertible matrix  $P$  such that  $P^{-1}AP$  is a diagonal matrix if  $A = \begin{bmatrix} 3 & 1 & 1 \\ -4 & -2 & -5 \\ 2 & 2 & 5 \end{bmatrix}$ .
3. Determine whether the following matrices are diagonalizable or not:  
(a)  $A = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$ ; (b)  $B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ ; (c)  $C = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$ .
4. Solve the following linear recurrences:
  - (a)  $x_{k+2} = 2x_k - x_{k+1}$ , where  $x_0 = 1$  and  $x_1 = 2$ ;
  - (b)  $x_{k+3} = -2x_k + x_{k+1} + 2x_{k+2}$ , where  $x_0 = 1$  and  $x_1 = 2 = x_2$ .

### Solutions

1.  $P^{-1} = \begin{bmatrix} 5 & 1 \\ 2 & 1 \end{bmatrix}^{-1} = \frac{1}{3} \begin{bmatrix} 1 & -1 \\ -2 & 5 \end{bmatrix}$ . So,  $P^{-1}AP = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} = \text{diag}(4, 1)$ . It follows that  $A = P\text{diag}(4, 1)P^{-1}$ . Therefore,

$$\begin{aligned} A^n &= P\text{diag}(4^n, 1)P^{-1} \\ &= \frac{1}{3} \begin{bmatrix} 5 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 4^n & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 5/3 4^n - 2/3 & -5/3 4^n + 5/3 \\ 2/3 4^n - 2/3 & -2/3 4^n + 5/3 \end{bmatrix}. \end{aligned}$$

2. The characteristic polynomial of  $A$  is

$$\det(xI_3 - A) = \begin{vmatrix} x-3 & -1 & -1 \\ 4 & x+2 & 5 \\ -2 & -2 & x-5 \end{vmatrix} = (x-1)(x-2)(x-3).$$

Thus  $A$  has three eigenvalues  $\lambda_1 = 1$ ,  $\lambda_2 = 2$  and  $\lambda_3 = 3$ . Eigenvectors: The homogeneous systems  $(I_3 - A)X = 0$ ,  $(2I_3 - A)X = 0$  and  $(3I_3 - A)X = 0$  have basic solutions  $X_1 = [1 \ -3 \ 1]^T$ ,  $X_2 = [1 \ -1 \ 0]^T$  and  $X_3 = [0 \ -1 \ 1]^T$ , respectively. There are basic eigenvectors corresponding to the respective eigenvalues. The matrix  $P = [X_1 \ X_2 \ X_3] =$

$$\begin{bmatrix} 1 & 1 & 0 \\ -3 & -1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \text{ diagonalizes } A \text{ with } P^{-1}AP = \text{diag}(1, 2, 3).$$

3. (a) Since the  $2 \times 2$  matrix  $A$  has two distinct (or simple, i.e., with multiplicity 1) eigenvalues  $\lambda_1 = -5$  and  $\lambda_2 = 2$ ,  $A$  is diagonalizable.
- (b) The matrix  $B$  has eigenvalue  $\lambda = -1$  with **multiplicity 2**. The matrix  $B$  is diagonalizable if there are two basic eigenvectors corresponding  $\lambda = -1$ . There are basic solutions to the homogeneous system  $(-I_3 - B)X = 0$ . So,  $X_1 = [-1 \ 1 \ 0]$  and  $X_2 = [-1, 0, 1]$  are **two basic eigenvectors** corresponding to  $\lambda = -1$ . The matrix  $B$  is diagonalizable.
- (c) The matrix  $C$  has an eigenvalue  $\lambda = 1$  of multiplicity 2. For  $C$  to be diagonalizable, there must be two basic eigenvectors corresponding to  $\lambda = 1$ . But the homogeneous system  $(I_3 - C)X = 0$

has only one basic solution  $X = [0 \ 1 \ 0]^T$ . Therefore,  $C$  is not diagonalizable.

4. (a) Define  $V_k = \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}$  for all  $k \geq 0$ . We have  $V_0 = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$  and  $V_{k+1} = \begin{bmatrix} x_{k+1} \\ x_{k+2} \end{bmatrix} = \begin{bmatrix} x_{k+1} \\ 2x_k - x_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} V_k$ . We obtain a linear dynamical system:  $V_k = A^k V_0$ .

Diagonalization of  $A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$ :  $c_A(x) = \det(xI_2 - A) =$

$$\begin{vmatrix} x & -1 \\ -2 & x+1 \end{vmatrix} = x(x+1) - 2 = x^2 + x - 2 = (x-1)(x+2).$$

$A$  has two simple eigenvalues  $\lambda_1 = -2$  and  $\lambda_2 = 1$ . Solving the homogeneous systems  $(-2I_2 - A)X = 0$  and  $(I_2 - A)X = 0$ , we find basic eigenvectors  $X_1 = [1 \ -2]^T$  and  $X_2 = [1 \ 1]^T$  corresponding to the eigenvalues  $\lambda_1 = -2$  and  $\lambda_2 = 1$ , respectively. So,

$P = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$ , with inverse  $P^{-1} = \frac{1}{3} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$  diagonalizes  $A$ .

$B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = P^{-1}V_0 = \frac{1}{3} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -1 \\ 4 \end{bmatrix}$ . It follows

that  $\begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} = V_k = b_1 \lambda_1^k X_1 + b_2 \lambda_2^k X_2 = \frac{-1}{3} (-2)^k X_1 + \frac{4}{3} X_2 = \frac{1}{3} \begin{bmatrix} -(-2)^k + 4 \\ 2(-2)^k + 4 \end{bmatrix}$ . Therefore,  $x_k = (4 - (-2)^k)/3$ .

- (b) Define  $V_k = \begin{bmatrix} x_k \\ x_{k+1} \\ x_{k+2} \end{bmatrix}$  for all  $k \geq 0$ . We have  $V_0 = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} =$

$$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \text{ and}$$

$$\begin{aligned} V_{k+1} &= \begin{bmatrix} x_{k+1} \\ x_{k+2} \\ x_{k+3} \end{bmatrix} = \begin{bmatrix} x_{k+1} \\ x_{k+2} \\ -2x_k + x_{k+1} + 2x_{k+2} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_k \\ x_{k+1} \\ x_{k+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{bmatrix} V_k. \end{aligned}$$

The matrix  $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{bmatrix}$  has eigenvalues  $\lambda_1 = 2$ ,  $\lambda_2 = -1$  and  $\lambda_3 = 1$  to which correspond the basic eigenvectors  $X_1 = [1 \ 2 \ 4]^T$ ,  $X_2 = [1 \ -1 \ 1]^T$  and  $X_3 = [1 \ 1 \ 1]^T$ , respectively. So,  $P = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 4 & 1 & 1 \end{bmatrix}$  with inverse  $P^{-1} = \frac{1}{6} \begin{bmatrix} -2 & 0 & 2 \\ 2 & -3 & 1 \\ 6 & 3 & -3 \end{bmatrix}$ , is a diagonalizing matrix for  $A$ . With  $B = [b_1 \ b_2 \ b_3]^T = P^{-1}V_0 = [1/3, -1/3, 1]$ , we have  $V_k = b_1\lambda_1^k X_1 + b_2\lambda_2^k X_2 + b_3\lambda_3^k X_3 = 2^k/3 X_1 - (-1)^k/3 X_2 + X_3 = \begin{bmatrix} 2^k/3 - (-1)^k/3 + 1 \\ 22^k/3 + (-1)^k/3 + 1 \\ 42^k/3 - (-1)^k/3 + 1 \end{bmatrix}$ . Therefore,  $x_k = 2^k/3 - (-1)^k/3 + 1$ .