

Practice Problems S7 (Vector Geometry)

- Let $P_1(2, 1, -2)$ and $P_2(1, -2, 0)$ be points in \mathbb{R}^3 .
 - Find the parametric equations of the line through P_1 and P_2 ;
 - Find the coordinates of the point P that is $1/4$ of the way from P_1 to P_2 .

- Find the point of intersection P between the lines (if they are concurrent):

$$\begin{cases} x = 3 + t \\ y = 2 + 3t \\ z = -1 - 3t \end{cases} \quad \text{and} \quad \begin{cases} x = 1 - s \\ y = 1 + 2s \\ z = 3 + s \end{cases} .$$

- Find the equation of the plane passing through the point $P(3, -7, 5)$ and is perpendicular to the line $\begin{cases} x = 2 + 6t \\ y = -5 - 6t \\ z = 3 + 5t \end{cases}$
- Find the equation of the plane through the points $A(3, -7, 1)$, $B(2, 0, -1)$ and $C(1, 3, 0)$. Check if the point $D(5, 1, 1)$ lies on this plane.

- Determine whether the plane $2x - 3y + z = 1$ contains the line $\begin{cases} x = 3 + 2t \\ y = 2 \\ z = 1 - 4t \end{cases}$.

- Find the line of intersection of the planes $(\pi_1) \equiv 3x + 5y + 4z = 5$ and $(\pi_2) \equiv x + 2y + 3z = 2$.

- Find the shortest distance from the point $P(1, 1, 1)$ to the line $\begin{cases} x = 3 + t \\ y = 9 \\ z = 10 - 4t \end{cases}$.

Which point on this line is closest to P ?

8. the shortest distance from the point $P(4, 1, 9)$ to the plane $x - 4z = 2$. Which point on this plane is closest to P ?
9. Find the areas of the sides of the parallelepiped determined by the vectors \overrightarrow{AB} , \overrightarrow{AC} , and \overrightarrow{AD} , where A, B, C and D are the points in Problem 4. What is the volume of this parallelepiped?

Recommended Problems:

Pages 165 - 167: 1 a, c; 3 a; 4 a; 5, 7 a; 9 a, b; 15, 20, 24

Pages 177 - 179: 1 a; 2 a, b; 3 a; 6, 8, 9, 10 a; 11 a; 12, 13, 14, 15, 16 a; 18, 19, 24 a

Page 185: 3 a; 4 a; 5 a.

Solutions

1. (a) If $P(x, y, z)$ is a point on the line through P_1 and P_2 , then there is $t \in \mathbb{R}$ such that $\overrightarrow{P_1P} = t\overrightarrow{P_1P_2} = t[-1 \ -3 \ 2]$. Thus,
$$\begin{cases} x = 2 - t \\ y = 1 - 3t \\ z = -2 + 2t \end{cases}.$$

(b) $\overrightarrow{P_1P} = \frac{1}{4}\overrightarrow{P_1P_2}$. This gives $P(7/4, 1/4, -3/2)$.

2. Solve the following system of linear equations
$$\begin{cases} x = 3 + t = 1 - s \\ y = 2 + 3t = 1 + 2s \\ z = -1 - 3t = 3 + s \end{cases}$$
 to get $t = -1 = s$. The point of intersection P has coordinates $P(2, -1, 2)$.

3. The plane has normal vector $\vec{v} = [6 \ -6 \ 5]^T$ (direction vector of the given line). So, the scalar equation is $6(x - 3) - 6(y + 7) + 5(z - 5) = 0$.
4. If $P(x, y, z)$ is a point on the plane through A , B and C , then the equation of the plane is

$$\begin{aligned} 0 &= \overrightarrow{AP} \cdot (\overrightarrow{AB} \times \overrightarrow{AC}) = \det(\overrightarrow{AP}, \overrightarrow{AB}, \overrightarrow{AC}) = \begin{vmatrix} x-3 & y+7 & z-1 \\ 2-3 & 0+7 & -1-1 \\ 1-3 & 3+7 & 0-1 \end{vmatrix} \\ &= 13(x-3) + 3(y+7) + 4(z-1). \end{aligned}$$

Replacing x , y and z by the coordinates of $D(5, 1, 1)$, we have: $13(5 - 3) + 3(1 + 7) + 4(1 - 1) = 26 + 24 = 50 \neq 0$. Therefore this plane does not contain the point D .

5. A line can be completely on the given plane, parallel to the plane or can intersect the plane at one point. Replace the parametric expressions of x , y and z into the equation of the plane. If one gets a consistent equation in the parameter t , then the line intersects the plane. If the parameter t is cancelled out, then the line is contained in the plane if the equation is consistent, otherwise, the line is parallel to the plane. With $x = 3 + 2t$, $y = 2$ and $z = 1 - 4t$, we have: $2(3 + 2t) - 3(2) + (1 - 4t) = 1$, i.e. $1 = 1$, this implies that the plane contains the line.

6. Solve $\begin{cases} 3x + 5y + 4z = 5 \\ x + 2y + 3z = 2 \end{cases}$ to find the line of intersection $\begin{cases} x = 7t \\ y = 1 - 5t \\ z = t \end{cases}$.
7. The line has direction vector $\vec{d} = [1 \ 0 \ -4]^T$. Choose arbitrarily a point $P_0 = (3, 9, 10)$ on the line. Project the vector $\overrightarrow{P_0P} = [-2 \ -8 \ -9]^T$ on \vec{d} : $\text{proj}_{\vec{d}}\overrightarrow{P_0P} = \frac{\overrightarrow{P_0P} \cdot \vec{d}}{\|\vec{d}\|^2}\vec{d} = \frac{-2 + 36}{17}\vec{d} = 2\vec{d}$. The shortest distance is $\|\overrightarrow{P_0P} - \text{proj}_{\vec{d}}\overrightarrow{P_0P}\| = \|\overrightarrow{P_0P} - 2\vec{d}\| = \sqrt{4^2 + 8^2 + 1^2} = 9$. The closest point $Q(x, y, z)$ is given by $\overrightarrow{P_0Q} = \text{proj}_{\vec{d}}\overrightarrow{P_0P}$. So, $Q(5, 9, 2)$.
8. The plane has normal vector $\vec{n} = [1 \ 0 \ -4]^T$. Choose arbitrarily a point $P_0(6, 0, 1)$ on the plane. Project the vector $\overrightarrow{P_0P} = [-2 \ 1 \ 8]^T$ on \vec{n} : The shortest distance is given by $\|\text{proj}_{\vec{n}}\overrightarrow{P_0P}\| = \|-2\vec{n}\| = 2\sqrt{17}$. The closest point $Q(x, y, z)$ is given by $\overrightarrow{QP} = \text{proj}_{\vec{n}}\overrightarrow{QP} = -2\vec{n}$. So, $Q(6, 1, 1)$.
9. The parallelepiped has six sides (2 times the parallelograms determined by the vectors). So, these sides have areas: $\|\overrightarrow{AB} \times \overrightarrow{AC}\| = \|[13 \ 3 \ 4]^T\| = \sqrt{194}$, $\|\overrightarrow{AB} \times \overrightarrow{AD}\| = \|[16 \ -4 \ -22]^T\| = \sqrt{756}$ and $\|\overrightarrow{AC} \times \overrightarrow{AD}\| = \|[8 \ -2 \ -36]^T\| = 2\sqrt{341}$. The volume of this parallelepiped is $Vol = |\overrightarrow{AD} \cdot (\overrightarrow{AB} \times \overrightarrow{AC})| = |\det(\overrightarrow{AD}, \overrightarrow{AB}, \overrightarrow{AC})| = 50$.