MATH 221 PRACTICE PROBLEMS

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1. Find A if: (a) $2A - \begin{bmatrix} 1 & -3 \end{bmatrix} = \left(\begin{bmatrix} 6 \\ 5 \end{bmatrix} - 3A^T \right)^T$; (b) $2A^T + \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix} = \left(\begin{bmatrix} 5 & 1 \\ -1 & 6 \end{bmatrix} - 3A \right)^T$

2. Find A in terms of B if: (a) $(2A - B)^T = A^T + (3B)^T$; (b) $(B^T - 3A)^T = 5A^T + 6B$

3. Show that every 1×3 matrix A can be written in the form
$$A = a \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$
for some scalars a, b and c. What can you say about 3×1 matrices?

4. If $A = -A$ where A is an $m \times n$ matrix, show that $A = 0$.

5. If A is a symmetric matrix, show that cA is also symmetric for any scalar c.

6. Show that $(-A)^T = -A^T$ for any matrix A.

7. If A and B are symmetric, show that $A - B$ is also symmetric.

8. A square matrix A is called **skew-symmetric** if $A^T = -A$.

   (a) Show that every 2×2 skew-symmetric matrix has the form $\begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$ for some scalar b.

   (b) If A and B are skew-symmetric, show that $A + B$ and cA are skew-symmetric for any scalar c.

9. Show that any square matrix A can be written in the form $A = S + W$ where S is symmetric and W is skew-symmetric. [*Hint*: First verify the identity $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$.]

10. In each case find the solution of the system whose augmented matrix has been carried to the following matrix R by row operations.

    (a) $R = \begin{bmatrix} 1 & 2 & 0 & 3 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (b) $R = \begin{bmatrix} 1 & 0 & 7 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -3 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

11. In each case find the rank of the given matrix, possibly in terms of the parameter a.

    (a) $\begin{bmatrix} 1 & -1 & 3 & 5 \\ 3 & -2 & 1 & -2 \\ -1 & 1 & 1 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -4 & -5 & 2 \\ 1 & 6 & 3 & 4 \\ 1 & 1 & -1 & 3 \end{bmatrix}$

    (c) $\begin{bmatrix} -1 & 3 & -2 & -2 \\ 3 & 1 & 1 & 9 \\ 1 & 7 & -3 & a \end{bmatrix}$ (d) $\begin{bmatrix} 1 & -2 & -5 & 3 \\ 2 & -3 & -8 & 7 \\ -2 & 4 & a+9 & a-7 \end{bmatrix}$

12. If a system of 5 equations in 7 variables has a solution, explain why there is more than one solution.

13. Suppose a system of 4 equations in 4 variables has a leading 1 in each row of the row-echelon form of its augmented matrix. Must there be a unique solution? Explain.

14. The graph of a linear equation $ax+by+cz=d$ is a plane in space. By examining the possible positions of three planes in space, explain geometrically why 3 equations in 3 variables must have zero, one or infinitely many solutions.

15. If A is carried to B by a row operation, show that B can be carried back to A by another row operation, and describe the new operation in terms of the original one.

16. Find a sequence of row operations carrying $\begin{bmatrix} b_1+c_1 & b_2+c_2 & b_3+c_3 \\ c_1+a_1 & c_2+a_2 & c_3+a_3 \\ a_1+b_1 & a_2+b_2 & a_3+b_3 \end{bmatrix} \to \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

17. The graph of the equation $x^2+y^2+ax+by+c=0$ is a circle for any choice of the numbers a, b and c. Find the circle through the three points $(1,2)$, $(3,-1)$ and $(0,-1)$.

18. Find the quadratic equation $f(x)=a+bx+cx^2$ which passes through the points $(0,1)$, $(1,2)$ and $(2,9)$. [This is called the **interpolating polynomial** for the three data points. It is used to find data points between given ones, and in plotting curves on computer monitors.]

19. In each case find all values of a for which the system has nontrivial solutions, and determine all solutions in each case.

(a) $\begin{array}{rcrcrcl} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ x_1 & + & ax_2 & - & 3x_3 & = & 0 \\ -x_1 & + & 6x_2 & - & 5x_3 & = & 0 \end{array}$

(b) $\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & x_3 & = & 0 \\ x_1 & + & 3x_2 & + & 6x_3 & = & 0 \\ 2x_1 & + & 3x_2 & + & ax_3 & = & 0 \end{array}$

(c) $\begin{array}{rcrcrcl} x_1 & + & x_2 & - & x_3 & = & 0 \\ & & ax_2 & - & 2x_3 & = & 0 \\ x_1 & + & x_2 & + & ax_3 & = & 0 \end{array}$

(d) $\begin{array}{rcrcrcl} ax_1 & + & x_2 & + & x_3 & = & 0 \\ x_1 & + & x_2 & - & x_3 & = & 0 \\ x_1 & + & x_2 & + & ax_3 & = & 0 \end{array}$

20. Consider the matrices $A=\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $B=\begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$, and $C=\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$.

(a) Show that the only choice of numbers x, y and z such that $xA+yB+zC=0$ is $x=y=z=0$. Because of this we say that the set $\{A, B, C\}$ of matrices is **linearly independent**.

(b) Is the set $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ linearly independent? Support your answer.

21. Show algebraically that there is a line through any two points in the plane. [*Hint*: Use the fact that every line has equation $ax+by+c=0$ where a, b and c are not all zero.]

22. Every plane in space has equation $ax+by+cz+d=0$ where a, b and c are not all zero. Show algebraically that there is a plane through any three points in space. [*Hint*: Preceding exercise.]

23. Find all solutions to the following system:
$$\begin{array}{rcrcrcr} x & + & y & + & 2z & = & 2 \\ 2x & + & y & - & z & = & 3 \\ x & + & 2y & + & 7z & = & 3 \end{array}$$

24. Find the augmented matrix, in reduced row-echelon form, of a system of equations in the variables x, y and z which has the following solutions: $x = 1 - 2t$, $y = -3 + t$ and $z = t$.

25. Find all solutions to the following system:
$$\begin{array}{rcrcrcr} x & + & 2y & + & z & = & -1 \\ 3x & + & 5y & + & z & = & 2 \\ -x & - & y & + & 3z & = & -5 \end{array}$$

26. Find all solutions to the following system:
$$\begin{array}{rcrcrcrcrcr} x_1 & - & x_2 & & & + & 2x_4 & + & x_5 & = & 2 \\ -2x_1 & + & 2x_2 & + & x_3 & - & 4x_4 & & & = & -7 \\ x_1 & - & x_2 & + & x_3 & + & 3x_4 & + & x_5 & = & -1 \end{array}$$

27. Find (if possible) conditions on the numbers a, b and c so that the following set of linear equations has no solution, a unique solution, or infinitely many solutions.
$$\begin{array}{rcrcrcr} x & - & y & + & 2z & = & a \\ 2x & - & y & + & 3z & = & b \\ -x & + & 2y & - & 3z & = & c \end{array}$$

28. Find conditions on a such that the system
$$\begin{array}{rcrcrcr} x & - & y & + & 2z & = & a \\ 2x & + & y & - & z & = & 3 \\ x & + & 5y & - & 8z & = & 1 \end{array}$$
has zero, one or infinitely many solutions.

29. Either prove the following statement or give an example showing that it it false: *If there is more than one solution to a system of linear equations, the augmented matrix A of the system has a row of zeros.*

30. Find all solutions to the system:
$$\begin{array}{rcrcrcrcrcr} x_1 & - & x_2 & + & 2x_3 & + & 2x_4 & + & 3x_5 & = & -4 \\ -2x_1 & + & 3x_2 & - & 6x_3 & - & 3x_4 & - & 11x_5 & = & 11 \\ -x_1 & + & 2x_2 & - & 4x_3 & + & x_4 & - & 8x_5 & = & 7 \\ & & x_2 & - & 2x_3 & + & 3x_4 & - & 5x_5 & = & 3 \end{array}$$

31. Find the augmented matrix, in reduced row-echelon form, of a system of three equations in five variables x_1, x_2, x_3, x_4, and x_5, with solutions $x_1 = 2t - s - 2$, $x_2 = 3$, $x_3 = s$, $x_4 = 6 - t$, and $x_5 = t$.

32. Simplify the following expressions where A, B and C represent matrices.
   (a) $A(3B - C) + (A - 2B)C + 2B(C + 2A)$
   (b) $A(B + C - D) + B(C - A + D) - (A + B)C + (A - B)D$
   (c) $AB(BC - CB) + (CA - AB)BC + CA(A - B)C$
   (d) $(A - B)(C - A) + (C - B)(A - C) + (C - A)^2$

33. If A is a real symmetric 2×2 matrix and $A^2 = 0$, show that $A = 0$. Give an example to show that it is essential that A is symmetric.

34. If $A = \begin{bmatrix} a & b & c \\ a_1 & b_1 & c_1 \end{bmatrix}$ and $AA^T = 0$, show that $A = 0$. [*Remark*: More generally, if A is *any* matrix such that $AA^T = 0$, then necessarily $A = 0$.]

35. If A is any matrix, show that AA^T is a symmetric matrix.

36. If A and B are matrices that both commute with a matrix C, show that the matrix $2A - 3B$ also commutes with C.

37. Find the matrix A if $\left[A^T - 3\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}\right]^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.

38. Find the matrix A if $[A - 2I]^{-1} = A^{-1}\begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix}$.

39. If A is a square matrix and $AX = 0$ for some matrix $X \neq 0$, show that A has no inverse.

40. If $U = \begin{bmatrix} 3 & -4 \\ 7 & 5 \end{bmatrix}$ and $AU = 0$ for some matrix A, show that necessarily $A = 0$.

41. If A and B are $n \times n$ matrices such that AB and B are both invertible, show that A is also invertible using *only* Theorem 3 §1.5.

42. If A and B are $n \times n$ matrices and $AB = cI$ where $c \neq 0$, show that $BA = cI$. Is it true if $c = 0$?

43. Let A be a square matrix which satisfies $A^3 - 2A^2 + 5A + 6I = 0$. Show that A is invertible, and find a formula for A^{-1} in terms of A.

44. If $E^2 = E$ and $A = I - 2E$, show that $A^{-1} = A$.

45. Find the inverse of $\begin{bmatrix} 1 & -1 & -1 \\ 2 & -1 & -4 \\ 1 & -2 & 2 \end{bmatrix}$.

46. If the first row of a square matrix A consists of zeros, show that A does not have an inverse.

47. If A is an invertible $n \times n$ matrix, show that $AX = B$ has a unique solution for any $n \times k$ matrix B.

48. If $det \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix} = 5$, find $det \begin{bmatrix} a+2x & b+2y & c+2z \\ x+p & y+q & z+r \\ 3p & 3q & 3r \end{bmatrix}$.

49. Find the values of the number c such that $\begin{bmatrix} 1 & c & 0 \\ 2 & 0 & c \\ c & -1 & 1 \end{bmatrix}$ has an inverse.

50. Find the inverse of $\begin{bmatrix} 1 & -1 & -2 \\ -1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$, and use it to solve $\left\{\begin{array}{rrrcl} x & -y & -2z & = & 3 \\ -x & & +z & = & 0 \\ 2x & +y & & = & 1 \end{array}\right.$

51. Assume that $det(A) = 3$ where $A = \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}$. Compute $det(-2B^{-1})$ where

$B = \begin{bmatrix} 2x & a+2p & p-3x \\ 2y & b+2q & q-3y \\ 2z & c+2r & r-3z \end{bmatrix}$.

52. Show that there is no real 3×3 matrix A such that $A^2 = -I$.

53. Show that $det \begin{bmatrix} p+x & q+y & r+z \\ a+x & b+y & c+z \\ a+p & b+q & c+r \end{bmatrix} = 2\, det \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}$.

54. Show that $det \begin{bmatrix} 1 & a & p & q \\ x & 1 & b & r \\ x^2 & x & 1 & c \\ x^3 & x^2 & x & 1 \end{bmatrix} = (1-ax)(1-bx)(1-cx)$ for any choice of p, q and r. [*Hint*: Begin by eliminating x from column 1.]

55. In each case evaluate $detA$ by inspection.

(a) $A = \begin{bmatrix} a & 3-a & a+1 \\ b & 3-b & b+1 \\ c & 3-c & c+1 \end{bmatrix}$ (b) $A = \begin{bmatrix} a & b & c \\ a+b & 2b & c+b \\ 3 & 3 & 3 \end{bmatrix}$

56. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} a+c & 2c \\ b+d & 2d \end{bmatrix}$. If $det\, A = 2$, find $det(A^2 B^T A^{-1})$.

57. Evaluate $det \begin{bmatrix} x-1 & 2 & 3 \\ 2 & -3 & x-2 \\ -2 & x & -2 \end{bmatrix}$ by first adding all other rows to the first row. Then find all values of x such that the determinant is zero.

58. If A is a 4×4 matrix and $A^2 = 3A$, what are the possible values of $det(A)$?

59. If $det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = -3$, compute $det \begin{bmatrix} 3 & -3 & 0 \\ c+5 & -5 & 3a \\ d-2 & 2 & 3b \end{bmatrix}$.

60. If A and B are $n \times n$ where n is odd, and if $AB = -BA$, show that either A or B has no inverse.

61. If A is 4×4 and $detA = 2$, find $det(15A^{-1} - 6\, adj A)$.

62. In each case: (1) Find the values of the number c such that A has an inverse, and (2) Find A^{-1} for those values of c.

(a) $A = \begin{bmatrix} c & c & 1 \\ 1 & c & 1 \\ c & -1 & 2 \end{bmatrix}$ (b) $A = \begin{bmatrix} 4 & c & 3 \\ c & 2 & c \\ 5 & c & 4 \end{bmatrix}$

63. If $detA = 3$, $detB = -1$ and $detC = 2$, compute the determinant of:

(a) $\begin{bmatrix} A & X & Y \\ 0 & B & Z \\ 0 & 0 & C \end{bmatrix}$ (b) $\begin{bmatrix} A & X & 0 \\ 0 & B & 0 \\ Y & Z & C \end{bmatrix}$

64. If A is 2×2 and B is 3×3, show that $det \begin{bmatrix} 0 & B \\ A & X \end{bmatrix} = detA\, detB$. [Hint: First left multiply by $\begin{bmatrix} 0 & I_2 \\ I_3 & 0 \end{bmatrix}$.]

65. Consider the matrix $A = \begin{bmatrix} 0 & 2 \\ 2 & -3 \end{bmatrix}$. Find the characteristic polynomial, eigenvalues and eigenvectors for A, and find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

66. Consider the matrix $A = \begin{bmatrix} -2 & 1 \\ 4 & 1 \end{bmatrix}$. Find the characteristic polynomial, eigenvalues and eigenvectors for A, and find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

67. Show that $A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$ is not diagonalizable.

68. If $A^k = 0$ for some $k \geq 1$, show that 0 is the only eigenvalue of A.

69. If A is a diagonalizable $n \times n$ matrix and every eigenvalue of A is zero, show that $A = 0$.

70. If $A^2 = A$, show that 0 and 1 are the only eigenvalues of A.

71. If A is a diagonalizable matrix, and if every eigenvalue λ of A satisfies $\lambda^2 = \lambda$, show that $A^2 = A$.

72. If A is a diagonalizable $n \times n$ matrix, show that A^2 is also diagonalizable.

73. If A is a diagonalizable $n \times n$ matrix, show that A^T is also diagonalizable.

74. Determine whether $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 2 \end{bmatrix}$ is diagonalizable.

75. Show that $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix}$ is not diagonalizable.

76. If A is diagonalizable and $\lambda_i \geq 0$ for each eigenvalue λ_i, show that $A = B^2$ for some matrix B. [*Hint*: If $P^{-1}AP = D = diag(\lambda_1, \cdots, \lambda_n)$, take $B = PD_0P^{-1}$ where $D_0 = diag(\sqrt{\lambda_1}, \cdots, \sqrt{\lambda_n})$.]

77. If A is diagonalizable and has only one eigenvalue λ, show that $A = \lambda I$.

78. If A is diagonalizable with eigenvalues λ_1, λ_2, $\cdots$ λ_n (possibly not all distinct), show that $detA = \lambda_1\lambda_2\cdots\lambda_n$. [*Remark*: This holds for any square matrix, diagonalizable or not.]

79. Find A^{-1} if $A = \begin{bmatrix} 1 & i \\ -i & 1+i \end{bmatrix}$.

80. Find a quadratic equation with real coefficients that has $2 - 3i$ as a root. What is the other root?

81. Show that $w = 3 - 2i$ is a root of $x^2 - 6x + 13$. What is the other root? Justify your answer.

82. Show that $z = (1+i)^n + (1-i)^n$ is a real number for each $n \geq 1$ by first finding the conjugate $\bar{z}$.

83. If $z \neq 0$ is a complex number, show that $1/z = \frac{1}{|z|^2}\bar{z}$.

84. If zw is real and $z \neq 0$, show that $w = r\,\bar{z}$ for some real number r.

85. Show that $|z+w|^2 + |z-w|^2 = 2(|z|^2 + |w|^2)$ for all complex numbers z and w. [*Hint*: $|z|^2 = z\bar{z}$.]

86. Find the point $\frac{1}{5}$ the way from $P(2, -1, 5)$ to $Q(3, 0, 4)$.

87. Find the two trisection points between $P(1, 2, 3)$ and $Q(8, -2, 0)$.

88. Let A, B and C denote the vertices of a triangle. If E is the midpoint of side BC, show that $\overrightarrow{AE} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC})$. [*Hint*: Start by writing $\overrightarrow{AE} = \overrightarrow{AB} + \overrightarrow{BE}$.]

89. The unit cube has three of its vertices $O(0, 0, 0)$, $A(1, 0, 0)$, $B(0, 1, 0)$ and $C(0, 0, 1)$. Show that, of the four diagonals of the unit cube, no two are perpendicular.

90. In each case write the vector $\vec{v}$ as a sum $\vec{v} = \vec{v}_1 + \vec{v}_2$ where $\vec{v}_1$ is parallel to $\vec{d}$ and $\vec{v}_2$ is orthogonal to $\vec{d}$. (a) $\vec{v} = [3\ -1\ 2]^T$ and $\vec{d} = [1\ 2\ 1]^T$. (b) $\vec{v} = [5\ 1\ -2]^T$ and $\vec{d} = [3\ 0\ -7]^T$.

91. If $\|\vec{v}\|^2 + \|\vec{w}\|^2 = \|\vec{v} + \vec{w}\|^2$ where $\vec{v} \neq \vec{0}$ and $\vec{w} \neq \vec{0}$, show that $\vec{v}$ and $\vec{w}$ are orthogonal.

92. Find the scalar equations of the line through the point $P(3, -1, 2)$ which is parallel to the line $[x\ y\ z]^T = [2 - 5t\ 3\ 2t]^T$ where t is arbitrary.

93. Find the scalar equations of the line through the points $P_1(1, 0, -2)$ and $P_2(2, 1, -1)$.

94. Find the point of intersection of the line $[x\ y\ z]^T = [2\ -1\ 3]^T + t[1\ -1\ -4]^T$ and the plane $3x + y - 2z = 4$.

95. Find the equation of the plane through the point $P(1, 1, -2)$ which contains the line $[x\ y\ z]^T = [3\ -1\ 0]^T + t[1\ 1\ -1]^T$.

96. Determine the equation of the line through the point $P(1, -1, 0)$ which is perpendicular to the plane $x + y - 2z = 3$.

97. Find the equation of the plane through the point $P_0(2, 3, -1)$ which is parallel to the plane with equation $4x - 3y + z = 4$.

98. Find the point Q on the line with equation $[x\ y\ z]^T = [1\ 2\ 0]^T + t[2\ -1\ 1]^T$ which is closest to the point $P(0, 1, 2)$.

99. Find the shortest distance from the point $P(1, 0, 2)$ to the plane $5x - 7y + 2z = 3$.

100. Find the shortest distance from the point $P(1, 0, 2)$ to the line $[x\ y\ z]^T = [1\ -1\ 0]^T + t[2\ 1\ 1]^T$.

101. Consider the the plane through the point $P_\circ(1, -1, 0)$ which is parallel to the plane with equation $2x - 3y + 2z = 4$. Does this plane pass through the origin? Support your answer.

102. Consider the points $A(2, 2, 1)$, $B(1, 1, 0)$ and $C(2, 3, -3)$.

    (a) Are these points the vertices of a right-angled triangle? Justify your answer.

    (b) Find the cosine of the interior angle of the triangle at vertex C.

103. Find the area of the triangle with vertices $A(1, 0, 0)$, $B(0, 1, 0)$ and $C(0, 0, 1)$.

104. Consider the transformation T defined as follows:

    Rotation through $\pi/2$ followed by reflection in the line $y = x$.

    Determine the effect of T, that is determine if it is a rotation (and find the angle) or a reflection or projection in some line through the origin (and find the line).

105. Find the reflection of the point $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ in the line $y = -3x$.

106. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation with $T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$ and $T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

    (a) Find the matrix of T and give a formula for $T\begin{bmatrix} x \\ y \end{bmatrix}$.

    (b) Compute $T^{-1}\begin{bmatrix} 2 \\ 2 \end{bmatrix}$.