FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS AND STATISTICS
FINAL EXAMINATION SOLUTION
MATH 221 (L05) FALL 2006

1. Solve the system:

$$\begin{array}{rcrcrcrcrcr} x & & & - & z & + & 2u & + & w & = & 2 \\ -2x & + & y & + & 2z & - & u & & & = & -7 \\ x & + & y & - & z & + & 3u & + & w & = & -1 \end{array}$$

Solution:

$$\left[\begin{array}{cccccc} 1 & 0 & -1 & 2 & 1 & 2 \\ -2 & 1 & 2 & -1 & 0 & -7 \\ 1 & 1 & -1 & 3 & 1 & -1 \end{array}\right] \begin{array}{c} \rightarrow \\ R_2+2R_1 \\ R_3-R_1 \end{array} \left[\begin{array}{cccccc} 1 & 0 & -1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 3 & 2 & -3 \\ 0 & 1 & 0 & 1 & 0 & -3 \end{array}\right] \begin{array}{c} \\ \rightarrow \\ R_3-R_2 \end{array} \left[\begin{array}{cccccc} 1 & 0 & -1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 3 & 2 & -3 \\ 0 & 0 & 0 & -2 & -2 & 0 \end{array}\right] \begin{array}{c} \\ \rightarrow \\ \left(-\frac{1}{2}\right) \end{array}$$

$$\text{Thus,}\quad \begin{array}{rcl} x & = & s+t+2 \\ y & = & t-3 \\ z & = & s \\ u & = & -t \\ w & = & t \end{array} \quad \text{where } s \text{ and } t \text{ are any numbers.}$$

2. Let $A = \left[\begin{array}{ccc} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{array}\right]$.

(a) Find all values of x so that A is not invertible.

Solution:

$$\det A = \left|\begin{array}{ccc} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{array}\right| \begin{array}{c} \\ R_2-xR_1 \\ R_3-xR_1 \end{array} = \left|\begin{array}{ccc} 1 & x & x \\ 0 & 1-x^2 & x-x^2 \\ 0 & x-x^2 & 1-x^2 \end{array}\right| = (1-x)^2 \left|\begin{array}{ccc} 1 & x & x \\ 0 & 1+x & x \\ 0 & x & 1+x \end{array}\right| = (1-x)^2 \left|\begin{array}{cc} 1+x & x \\ x & 1+x \end{array}\right| =$$

$(1-x)^2(2x+1)$.

A is not invertible exactly when $\det A = (1-x)^2(2x+1) = 0$, that is, $x = 1$ or $x = -\frac{1}{2}$

(b) Is it true that if A is not invertible then the system $AX = 0$ has no solutions? Explain.

Solution: It is not true that if A is not invertible then the system $AX = 0$ has no solutions, because for any matrix A the homogeneous system $AX = 0$ always has a solution, namely, $X = 0$.

3. Let $A = \left[\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 1 & 3 \end{array}\right]$ and $B = \left[\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right]$.

(a) Find an invertible matrix U such that $UA = B$.

Solution:

$$\left[\begin{array}{ccc|cc} 1 & 2 & 3 & 1 & 0 \\ 2 & 1 & 3 & 0 & 1 \end{array}\right] \begin{array}{c} E_1 \\ R_2-2R_1 \end{array} \left[\begin{array}{ccc|cc} 1 & 2 & 3 & 1 & 0 \\ 0 & -3 & -3 & -2 & 1 \end{array}\right] \begin{array}{c} \rightarrow \\ -\frac{1}{3}R_2 \end{array} \left[\begin{array}{ccc|cc} 1 & 2 & 3 & 1 & 0 \\ 0 & 1 & 1 & \frac{2}{3} & \frac{-1}{3} \end{array}\right] \begin{array}{c} R_1-2R_2 \\ \rightarrow \end{array} \left[\begin{array}{ccc|} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right.$$

Thus, $U = \left[\begin{array}{cc} \frac{-1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{-1}{3} \end{array}\right]$

(b) Express U^{-1} as a product of elementary matrices.

$U = E_3E_2E_1$, so $U^{-1} = (E_3E_2E_1)^{-1} = E_1^{-1}E_2^{-1}E_3^{-1} = \left[\begin{array}{cc} 1 & 0 \\ 2 & 1 \end{array}\right]\left[\begin{array}{cc} 1 & 0 \\ 0 & -3 \end{array}\right]\left[\begin{array}{cc} 1 & 2 \\ 0 & 1 \end{array}\right]$.

4. Let $A = \left[\begin{array}{cc} 3 & -4 \\ 1 & -2 \end{array}\right]$.

(a) Find an invertible matrix P and a diagonal matrix D so that $P^{-1}AP = D$.

$$c_A(x) = \det(A - xI) = \left|\begin{array}{cc} 3-x & -4 \\ 1 & -2-x \end{array}\right| = (3-x)(-2-x)+4 = x^2-x-2 = (x+1)(x-2) = 0$$

when $x = -1$ or $x = 2$.

Thus, the eigenvalues are $\lambda_1 =$ and $\lambda_2 = 2$.

To find the eigenvectors corresponding to the eigenvalue -1, we solve the system $(A+I)X = 0$

$$\left[\begin{array}{ccc} 4 & -4 & 0 \\ 1 & -1 & 0 \end{array}\right] \begin{array}{c} R_1-4R_2 \\ \rightarrow \end{array} \left[\begin{array}{ccc} 0 & 0 & 0 \\ 1 & -1 & 0 \end{array}\right] \begin{array}{c} R_1 \longleftrightarrow R_2 \\ \rightarrow \end{array} \rightarrow \left[\begin{array}{ccc} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

The eigenvectors corresponding to the eigenvalue -1 are $X = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ where t is any number.
To find the eigenvectors corresponding to the eigenvalue 2, we solve the system $(A-2I)X=0$

$$\begin{bmatrix} 1 & -4 & 0 \\ 1 & -4 & 0 \end{bmatrix} \underset{R_2-R_1}{\rightarrow} \begin{bmatrix} 1 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The eigenvectors corresponding to the eigenvalue 2 are $X = t\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ where t is any number.

$P = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$

(b) Compute A^7.
Solution:

$$\begin{aligned} A^7 = &= PD^7P^{-1} &= \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} (-1)^7 & 0 \\ 0 & 2^7 \end{bmatrix}\left(\frac{1}{3}\begin{bmatrix} -1 & 4 \\ 1 & -1 \end{bmatrix}\right) \\ && = \frac{1}{3}\begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 128 \end{bmatrix}\begin{bmatrix} -1 & 4 \\ 1 & -1 \end{bmatrix} \\ && = \frac{1}{3}\begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -4 \\ 128 & -128 \end{bmatrix} \\ && = \frac{1}{3}\begin{bmatrix} 513 & -516 \\ 129 & -132 \end{bmatrix} \\ && = \begin{bmatrix} 171 & -172 \\ 43 & -44 \end{bmatrix} \end{aligned}$$

5. Let $A^{-1} = \begin{bmatrix} 2 & 1 & 2 \\ -3 & -1 & -1 \\ 5 & 2 & 1 \end{bmatrix}$.

(a) Find $\det A$.

Solution: $\det A^{-1} = \begin{vmatrix} 2 & 1 & 2 \\ -3 & -1 & -1 \\ 5 & 2 & 1 \end{vmatrix} \begin{matrix} R_1+R_2 \\ \\ R_3+2R_2 \end{matrix} = \begin{vmatrix} -1 & 0 & 1 \\ -3 & -1 & -1 \\ -1 & 0 & -1 \end{vmatrix} = -\begin{vmatrix} -1 & 1 \\ -1 & -1 \end{vmatrix} = -2$, and so $\det A = \dfrac{1}{\det A^{-1}} = -\dfrac{1}{2}$.

(b) Find $\det\left(A^{-1} + 2adjA\right)$.

Solution: $\det\left(A^{-1}+2adjA\right) = \det\left(A^{-1}+2(\det A)A^{-1}\right) = \det\left(A^{-1}+2\left(-\frac{1}{2}\right)A^{-1}\right) = \det 0 = 0.$

6. Let A be a square matrix. Prove the following statements:
(a) If A is not invertible then 0 is an eigenvalue of A.
Solution: Suppose that A is not invertible, then the homogeneous system $AX=0$ has a non-trivial solution, that is there exist a nonzero column X so that $AX = 0 = 0X$, which implies that 0 is an eigenvalue of A.
(b) If A is diagonalizable then A^T is also diagonalizable.
Solution: Suppose that A is diagonalizable. Then there exists an invertible matrix P and a diagonal matrix D so that $P^{-1}AP = D$. Now, $P^TA^T\left(P^{-1}\right)^T = \left(P^{-1}AP\right)^T = D^T = D$. Thus,

$$P^TA^T\left(P^{-1}\right)^T = D. \qquad (*)$$

Let $Q = \left(P^{-1}\right)^T = \left(P^T\right)^{-1}$. then $Q^{-1} = P^T$ and $(*)$ becomes $Q^{-1}A^TQ = D$ which implies that A^T is diagonalizable.

7. For the following, express your answers in the form $a+bi$ where a and b are real numbers.
(a) Compute $\left(1-\sqrt{3}i\right)^{10}$.
Solution:

$$
\begin{aligned}
\left(1-\sqrt{3}i\right)^{10} &= \left(2e^{i\left(-\frac{\pi}{3}\right)}\right)^{10} \\
&= 2^{10}e^{i\left(-\frac{10\pi}{3}\right)} \\
&= 2^{10}\left(\cos\left(-\frac{10\pi}{3}\right)+i\sin\left(-\frac{10\pi}{3}\right)\right) \\
&= 2^{10}\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) \\
&= -512+512\sqrt{3}i
\end{aligned}
$$

(b) Find all complex numbers z so that $z^4 = -16$.

8. Consider the points $A\,(2,1,-2)$, $B\,(4,1,0)$ and $C\,(6,3,0)$.
(a) Find the internal angles of the triangle with vertices A, B and C.
Solution: Let α,β,γ be the angles at A,B,C respectively.

Since α is the angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$, $\cos\alpha = \frac{\overrightarrow{AB}\bullet\overrightarrow{AC}}{\left\|\overrightarrow{AB}\right\|\left\|\overrightarrow{AC}\right\|} = \frac{[2,0,2]^T\bullet[4,2,2]^T}{\sqrt{8}\sqrt{24}} = \frac{12}{8\sqrt{3}} = \frac{\sqrt{3}}{2}$ and so $\alpha = \dfrac{\pi}{6}$.

Similarly, β is the angle between $\overrightarrow{BA}$ and $\overrightarrow{BC}$, $\cos\beta = \frac{\overrightarrow{BA}\bullet\overrightarrow{BC}}{\left\|\overrightarrow{BA}\right\|\left\|\overrightarrow{BC}\right\|} = \frac{[-2,0,-2]^T\bullet[2,2,0]^T}{\sqrt{8}\sqrt{8}} = \frac{-4}{8} = \frac{-1}{2}$ and so $\beta = \dfrac{2\pi}{3}$.

Lastly, $\gamma = \pi-(\alpha+\beta) = \pi - \left(\dfrac{\pi}{6}+\dfrac{2\pi}{3}\right) = \dfrac{\pi}{6}$

(b) Find an equation of the plane containing the points A,B and C.
Solution: A normal of the plane is $\overrightarrow{n} = \frac{1}{4}\overrightarrow{AB}\times\overrightarrow{AC} = \frac{1}{4}[2,0,2]^T\times[4,2,2]^T = [1,0,1]^T\times[2,1,1]^T = [-1,1,1]^T$ and so an equation of the plane is $-x+y+z=-3$.

9. Let P_1 be the plane with equation $x+2y-z=2$ and P_2 be the plane with equation $2x-y+z=2$. Let L be the line of intersection of the planes P_1 and P_2.
(a) Is the point $A(1,1,1)$ on both of the planes P_1 and P_2? Explain.
Solution: Yes, the point $A(1,1,1)$ is on both of the planes P_1 and P_2 because its coordinates satisfy both equations of the two planes.
(b) Find an equation of the line L.
Solution: Since L lies in both planes, it is perpendicular to both of the normals $\overrightarrow{n_1}$ and $\overrightarrow{n_2}$ of the planes, we can choose a direction of the line to be $\overrightarrow{d} = \overrightarrow{n_1}\times\overrightarrow{n_2} = [1,2,-1]^T\times[2,-1,1]^T = [1,-3,-5]^T$, and from part (a), a pont on L is $A(1,1,1).Thus$ an equation of L is $[x,y,z]^T = [1,1,1]^T + t\,[1,-3,-5]^T$
(c) Find the shortest distance between the point $B\,(4,-3,-3)$ and the line L, also find the point Q on the line L that is closest to B.
Solution: Q on the line L, so the coordinates of Q is $Q\,(1+t,1-3t,1-5t)$ and therefore, $\overrightarrow{BQ} = [t-3,4-3t,4-5t]^T$. Since Q is the point on the line L that is closest to B, we have $\overrightarrow{BQ}\bullet\overrightarrow{d}=0$, that is, $(t-3)-3\,(4-3t)-5\,(4-5t)=0$ which gives $t=1$. Thus the coordinates of Q is $Q\,(2,-2,-4)$ and the shortest distance between the point $B\,(4,-3,-3)$ and the line L is $\left\|\overrightarrow{BQ}\right\| = \left\|[-2,1,-1]^T\right\| = \sqrt{6}$.

10 . Let $T:\mathbb{R}^2\to\mathbb{R}^2$ be a linear transformation such that $T\left[\begin{smallmatrix}2\\1\end{smallmatrix}\right] = \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]$ and $T\left[\begin{smallmatrix}3\\2\end{smallmatrix}\right] = \left[\begin{smallmatrix}2\\1\end{smallmatrix}\right]$.
(a)Find the matrix of T; that is, find a matrix A so that $T\overrightarrow{v} = A\overrightarrow{v}$ for all $\overrightarrow{v}\in\mathbb{R}^2$.
Solution: Since $T\left[\begin{smallmatrix}2\\1\end{smallmatrix}\right] = A\left[\begin{smallmatrix}2\\1\end{smallmatrix}\right] = \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]$ and $T\left[\begin{smallmatrix}3\\2\end{smallmatrix}\right] = A\left[\begin{smallmatrix}3\\2\end{smallmatrix}\right] = \left[\begin{smallmatrix}2\\1\end{smallmatrix}\right]$, we get $A\begin{bmatrix}2&3\\1&2\end{bmatrix} = \begin{bmatrix}1&2\\2&1\end{bmatrix}$, and so,

$$A = \begin{bmatrix}1&2\\2&1\end{bmatrix}\begin{bmatrix}2&3\\1&2\end{bmatrix}^{-1} = \begin{bmatrix}1&2\\2&1\end{bmatrix}\begin{bmatrix}2&-3\\-1&2\end{bmatrix} = \begin{bmatrix}0&1\\3&-4\end{bmatrix}$$

(b) Is T invertible? If T is invertible, find the matrix of T^{-1}.
Solution: T is invertible because its matrix is invertible (note that $\det A=-3$), and the matrix of T^{-1} is $A^{-1} = \frac{1}{3}\begin{bmatrix}4&1\\3&0\end{bmatrix}$
(c) Is there a vector $\overrightarrow{a}\in\mathbb{R}^2$ so that $T\overrightarrow{a} = \left[\begin{smallmatrix}-3\\7\end{smallmatrix}\right]$? If so, find $\overrightarrow{a}$.
Solution: Yes, in fact, $\overrightarrow{a} = T^{-1}\left(T\overrightarrow{a}\right) = T^{-1} = A^{-1}\left[\begin{smallmatrix}-3\\7\end{smallmatrix}\right] = \frac{1}{3}\begin{bmatrix}4&1\\3&0\end{bmatrix}\begin{bmatrix}-3\\7\end{bmatrix} = \frac{1}{3}\begin{bmatrix}-5\\-9\end{bmatrix}$

End of Examination