

Practice Problems S1

1. Consider the following system of linear equations in x and y

$$\begin{cases} x + ay = 1 \\ ax + 4y = 2 \end{cases},$$

where $a \in \mathbb{R}$ is a parameter.

- write down the coefficient matrix and the augmented matrix of the system;
- Find all values of (if any) for which the system has no solution, exactly one solution or infinitely many solutions.

2. Let

$$\begin{bmatrix} 1 & -1 & 3 & 5 \\ 3 & -2 & 1 & -2 \\ -1 & 1 & 1 & 3 \end{bmatrix}.$$

- Carry the matrix A to a row-echelon form;
- Find the rank of A ;
- Use the row echelon form of A from part (a) to solve the system

$$\begin{cases} x_1 - x_2 + 3x_3 = 5 \\ 3x_1 - 2x_2 + x_3 = -2 \\ -x_1 + x_2 + x_3 = 3 \end{cases}.$$

3. (a) Find the reduced row-echelon form of the matrix

$$\begin{bmatrix} 1 & -1 & 3 & 1 & 3 \\ -1 & -2 & 6 & 2 & -6 \\ 2 & 1 & 3 & 5 & 3 \\ 2 & -2 & 12 & 8 & 0 \end{bmatrix}.$$

(b) Solve the system

$$\begin{cases} x_1 - x_2 + 3x_3 + x_4 = 3 \\ -x_1 - 2x_2 + 6x_3 + 2x_4 = -6 \\ 2x_1 + x_2 + 3x_3 + 5x_4 = 3 \\ 2x_1 - 2x_2 + 12x_3 + 8x_4 = 0 \end{cases}$$

4. Find basic solutions of the following homogeneous system:

$$\begin{cases} x_1 - x_2 + 3x_3 + x_4 = 0 \\ -x_1 - 2x_2 + 6x_3 + 2x_4 = 0 \\ 2x_1 + x_2 + 3x_3 + 5x_4 = 0 \\ 2x_1 - 2x_2 + 12x_3 + 8x_4 = 0 \end{cases}$$

Recommended Problems:

Pages: 7 - 8: 1, 7, 8, 9, 10, 12

15 - 16: 1; 2(a); 3(a); 5(a), (b); 6(b); 7(a), (b); 8; 9(b); 11.

Solutions

1. (a) $\begin{cases} x+ay=1 \\ ax+4y=2 \end{cases}$ has coefficient matrix
 $\begin{bmatrix} 1 & a \\ a & 4 \end{bmatrix}$ and augmented matrix $\begin{bmatrix} 1 & a & | & 1 \\ a & 4 & | & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & a & | & 1 \\ a & 4 & | & 2 \end{bmatrix} \xrightarrow{R_2 - aR_1} \begin{bmatrix} 1 & a & | & 1 \\ 0 & 4-a^2 & | & 2-a \end{bmatrix}$

Case 1: If $4-a^2 \neq 0$, i.e. $a \neq \pm 2$

$\xrightarrow{R_2(4-a^2)} \begin{bmatrix} 1 & a & | & 1 \\ 0 & 1 & | & \frac{1}{2+a} \end{bmatrix}$

In this case $y = \frac{1}{2+a}$ and $x = 1-ay$
 $= 1 - \frac{a}{2+a} = \frac{2}{2+a}$

i.e. the system has exactly one solution

$x = \frac{2}{2+a}$ and $y = \frac{1}{2+a}$ if $a \neq 2$ and $a \neq -2$.

Case 2: If $a^2 - 4 = 0$, then $a=2$ or $a=-2$.

For $\boxed{a=2}$, we have

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & 0 \end{array} \right]$$

The system has infinitely many solutions;

$$y=t \Rightarrow x=1-2t \quad \text{for } t \in \mathbb{R}.$$

For $\boxed{a=-2}$, we have

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 0 & 4 \end{array} \right]$$

The system is inconsistent, i.e. has no solution.

2. (a) $\left[\begin{array}{cccc} 1 & -1 & 3 & 5 \\ 3 & -2 & 1 & -2 \\ -1 & 1 & 1 & 3 \end{array} \right] \xrightarrow{R_2 \rightarrow R_2 - 3R_1} \left[\begin{array}{cccc} 1 & -1 & 3 & 5 \\ 0 & 1 & -8 & -17 \\ -1 & 1 & 1 & 3 \end{array} \right] \xrightarrow{R_3 \rightarrow R_3 + R_1} \left[\begin{array}{cccc} 1 & -1 & 3 & 5 \\ 0 & 1 & -8 & -17 \\ 0 & 0 & -4 & 8 \end{array} \right]$

$$\xrightarrow{\quad} \left[\begin{array}{cccc} 1 & -1 & 3 & 5 \\ 0 & 1 & -8 & -17 \\ 0 & 0 & \cancel{-4} & \cancel{8} \end{array} \right] \rightarrow \text{is a row-echelon matrix}$$

(b) This row-echelon matrix has 3 leading 1's.
Therefore the rank of A is 3.

(c) The system is equivalent to

$$x_1 - x_2 + 3x_3 = 5$$

$$x_2 - 8x_3 = -17$$

$$x_3 = 2$$

Back-substitution: $x_3 = 2$

$$\Rightarrow x_2 - 16 = -17 \Rightarrow x_2 = -1$$

$$\Rightarrow x_1 + 1 + 6 = 5 \Rightarrow x_1 = -2$$

So, the system has exactly one solution

$$x_1 = -2, x_2 = 2, x_3 = -1.$$

$$3. (a) \begin{bmatrix} 1 & -1 & 3 & 1 & 3 \\ -1 & -2 & 6 & 2 & -6 \\ 2 & 1 & 3 & 5 & 3 \\ 2 & -2 & 12 & 8 & 0 \end{bmatrix} \xrightarrow{\begin{array}{l} R_2 \leftrightarrow R_2 + R_1 \\ R_3 \leftrightarrow R_3 - 2R_1 \\ R_4 \leftrightarrow R_4 - 2R_1 \end{array}} \begin{bmatrix} 1 & -1 & 3 & 1 & 3 \\ 0 & -3 & 9 & 3 & -3 \\ 0 & 3 & -3 & 3 & -3 \\ 0 & 6 & 6 & 6 & -6 \end{bmatrix}$$

$$\xrightarrow{\begin{array}{l} R_3 \leftrightarrow R_3/3 \\ R_2 \leftrightarrow R_2/3 \\ R_4 \leftrightarrow R_4/6 \end{array}} \begin{bmatrix} 1 & -1 & 3 & 1 & 3 \\ 0 & 1 & -3 & -1 & 1 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & 1 & 1 & -1 \end{bmatrix} \xrightarrow{\begin{array}{l} R_3 \leftrightarrow R_3 - R_1 \\ R_4 \leftrightarrow R_4 - R_1 \end{array}} \begin{bmatrix} 1 & -1 & 3 & 1 & 3 \\ 0 & 1 & -3 & -1 & 1 \\ 0 & 0 & 2 & 2 & -2 \\ 0 & 0 & 1 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow{\begin{array}{l} R_3 \leftrightarrow R_3/2 \\ R_4 \leftrightarrow R_4 - \frac{1}{2}R_3 \end{array}} \begin{bmatrix} 1 & -1 & 3 & 1 & 3 \\ 0 & 1 & -3 & -1 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is a row-echelon matrix

$$\left[\begin{array}{cccc|c} 1 & -1 & 3 & 1 & 3 \\ 0 & 1 & -3 & -1 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{array} \right] \xrightarrow{\begin{matrix} R_1 + R_2 \\ R_2 + 3R_3 \end{matrix}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 2 & -2 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{array} \right]$$

So, $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 2 & -2 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$ is the reduced row-echelon matrix of A

(b) Since the augmented matrix of the system has exactly the reduced row-echelon matrix from part (a), the system is equivalent

to

$$\begin{cases} x_1 = 4 \\ x_2 + 2x_4 = -2 \\ x_3 + x_4 = -1 \end{cases} \quad \text{Set } x_4 = t$$

The system has infinitely many solutions

$$x_1 = 4$$

$$x_2 = -2 - 2t$$

$$x_3 = -1 - t \quad \text{for } t \in \mathbb{R}$$

$$x_4 = t$$

(4) The coefficient matrix of the homogeneous system is

$$\begin{bmatrix} 1 & -1 & 3 & 1 \\ -1 & -2 & 6 & 2 \\ \frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{5}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{12}{2} & \frac{5}{2} \end{bmatrix}$$

with reduced row-echelon matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

It has rank 3.

The system has 1 basic solution.

$$x_1 = 0$$

$$x_2 = -2t$$

$$x_3 = -t, t \in \mathbb{R} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = t \begin{bmatrix} 0 \\ -2 \\ -1 \\ 1 \end{bmatrix}, t \in \mathbb{R}$$

$$x_4 = t$$

is the general solution

$$\begin{bmatrix} 0 \\ -2 \\ -1 \\ 1 \end{bmatrix}$$

is a basic solution.