Practice Problems S3

1. Find the matrix of the reflection in the line $y = -x$.

2. Find a rotation or a reflection that is equal to

   (a) reflection in the y-axis followed by rotation through $\pi/2$;

   (b) rotation through $\pi/2$ followed by reflection in the line $y = x$.

3. Given $T([1 \ -2]^T) = [3 \quad 4]^T$ and $T([-2 \quad 5]^T) = [-1 \quad 4]^T$, find $T([-4 \quad 3]^T)$ if T is a linear transformation.

4. Consider a Markov chain that starts in state 1 with transition matrix $P = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$.

   (a) Explain why the chain is regular.

   (b) Find the probability that the chain is in state 1 after 2 transitions.

   (c) Find the steady-state vector for the chain.

5. Find the inverse of $A = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$ using the adjoint formula.

6. Given $A = \begin{bmatrix} 3 & -1 & 2 \\ 5 & 5 & -2 \\ 1 & 2 & 3 \end{bmatrix}$, find the $(1,3)$-entry of A^{-1}.

7. For which values of $c \in \mathbb{R}$ is A invertible if $A = \begin{bmatrix} 1 & c & 0 \\ 2 & 0 & c \\ c & -1 & 1 \end{bmatrix}$.

8. Solve the following system by Cramer's rule:

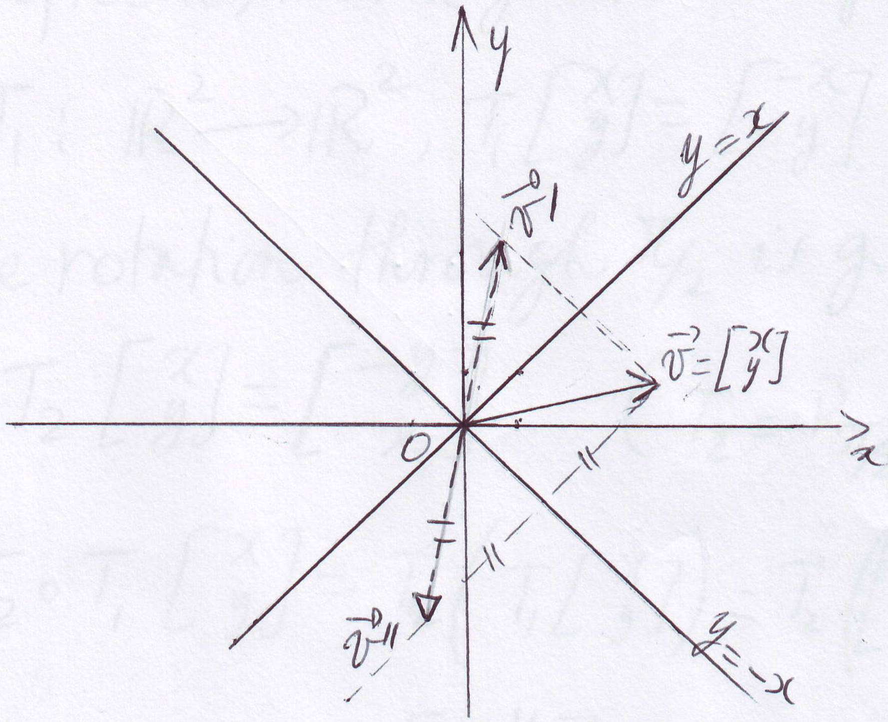
$$\begin{cases} x + 2y = 4 \\ 3x + 7y = 13 \end{cases} .$$

Recommended Problems:

Pages 80-81: 1. b, c; 2. a; 3, 4, 5, 9, 10, 12; Pages 101-102: 1, 2, a, c; Page 11: 1. a, b, f, g, h, k, l, m, n, o, p; 5. a, b; 6, 7, 8, 9, 11, 13, 14, 15; Page 126: 1. a, c; 2. a, b, c, d; 3, 4, 6, 8, 9, 10.

Solutions

1.



Denote by $\vec{v}''$ the reflection of $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ in the line $y = -x$ and by $\vec{v}'$ in the line $y = x$. We know that $\vec{v}' = \begin{bmatrix} y \\ x \end{bmatrix}$. From the figure, we see that $\vec{v}''$ is the reflection of $\vec{v}'$ about the origin. Therefore $\vec{v}'' = \begin{bmatrix} -y \\ -x \end{bmatrix}$. It follows that it is a linear transformation with matrix $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$. It is the composition of the reflection in the line $y = x$ followed by the reflection about the origin: $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$.

2. a) Reflection in the y-axis is defined by

$$T_1 : \mathbb{R}^2 \to \mathbb{R}^2, \; T_1 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$

The rotation through $\pi/2$ is given by

$$T_2 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix} \cdot \; (T_2 = R_{\pi/2})$$

$$T_2 \circ T_1 \begin{bmatrix} x \\ y \end{bmatrix} = T_2\left(T_1 \begin{bmatrix} x \\ y \end{bmatrix} \right) = T_2\left(\begin{bmatrix} -x \\ y \end{bmatrix} \right)$$

$$= \begin{bmatrix} -y \\ -x \end{bmatrix}$$

Therefore, the reflection in the y-axis followed by the rotation through $\frac{\pi}{2}$ is the reflection in the line $y = -x$

(b) $R_{\pi/2}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -y \\ x \end{bmatrix}$

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ x \end{bmatrix}$ is the reflection about $x = y$

$$T \circ R_{\pi/2}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = T\left(R_{\pi/2}\begin{bmatrix} x \\ y \end{bmatrix}\right) = T\left(\begin{bmatrix} -y \\ x \end{bmatrix}\right) = \begin{bmatrix} x \\ -y \end{bmatrix}$$

It is the reflection in the x-axis.

3. $T\left(\begin{bmatrix}1\\-2\end{bmatrix}\right)=\begin{bmatrix}3\\4\end{bmatrix}$, $T\left(\begin{bmatrix}-2\\5\end{bmatrix}\right)=\begin{bmatrix}-1\\4\end{bmatrix}$

$T\left(\begin{bmatrix}-4\\3\end{bmatrix}\right)$ can be computed if we can express $\begin{bmatrix}-4\\3\end{bmatrix}$ as a linear combination of vectors $\begin{bmatrix}1\\-2\end{bmatrix}$ and $\begin{bmatrix}-2\\5\end{bmatrix}$; i.e., find a and b such that $\begin{bmatrix}-4\\3\end{bmatrix}=a\begin{bmatrix}1\\-2\end{bmatrix}+b\begin{bmatrix}-2\\5\end{bmatrix}$

$$\begin{cases}-4 = a-2b\\ 3 = -2a+5b\end{cases} \Rightarrow \begin{bmatrix}1 & -2\\-2 & 5\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}-4\\3\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}1 & -2\\-2 & 5\end{bmatrix}^{-1}\begin{bmatrix}-4\\3\end{bmatrix}=\frac{1}{5-4}\begin{bmatrix}5 & 2\\2 & 1\end{bmatrix}\begin{bmatrix}-4\\3\end{bmatrix}$$

$$=\begin{bmatrix}-14\\-5\end{bmatrix}$$, i.e., $a=-14$, $b=-5$

We have, $T\left(\begin{bmatrix}-4\\3\end{bmatrix}\right)=T\left(-14\begin{bmatrix}1\\-2\end{bmatrix}-5\begin{bmatrix}-2\\5\end{bmatrix}\right)$

$$=-14\,T\begin{bmatrix}1\\-2\end{bmatrix}-5\,T\begin{bmatrix}-2\\5\end{bmatrix}$$

$$=-14\begin{bmatrix}3\\4\end{bmatrix}-5\begin{bmatrix}-1\\4\end{bmatrix}=\begin{bmatrix}-42+5\\-56-20\end{bmatrix}$$

$$=\begin{bmatrix}-37\\-76\end{bmatrix}.$$

4. $P = \begin{bmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix}$, $S_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (starting in State 1)

(a) Since $P^1 = P$ has only positive numbers, all powers of P will also have positive entries. It follows that this chain is regular

(b) $S_{m+1} = P S_m$, $S_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
$= P^{m+1} S_0$

$$S_1 = P S_0 = \begin{bmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$$

$$S_2 = P S_1 = \begin{bmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 5/9 \\ 4/9 \end{bmatrix}$$

is the state vector after two transitions.

So, the probability for the chain to be in State 1 after 2 transitions is 5/9 (in state 2 it is 4/9)

(c) Since the chain is regular, it has a steady-state vector S which is a probability vector solving the homogeneous system $(I_2 - P)S = 0$

$$I_2 - P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 2/3 & -2/3 \\ -2/3 & 2/3 \end{bmatrix}$$

$$\begin{bmatrix} 2/3 & -2/3 \\ -2/3 & 2/3 \end{bmatrix} \xrightarrow[r_2 \to r_2 + r_1]{r_1 \to \frac{3}{2} r_1} \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \end{array}$$

$$\Rightarrow \quad x_1 - x_2 = 0 \quad \Rightarrow \quad x_1 = x_2 = t$$

$$X = \begin{bmatrix} t \\ t \end{bmatrix}$$

X is a probabilty vector iff $0 \le t \le 1$ and $t + t = 1$

$$\Rightarrow S = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

5. $\det A = \begin{vmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & -1 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} = -1$

$$A^{-1} = \frac{1}{\det A} \operatorname{adj}(A) = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

6. The $(1,3)$-entry of A^{-1} is

$$\frac{1}{\det A} \operatorname{adj}(A)_{13} = \frac{1}{\det A} \cdot C_{31}(A) = \frac{1}{84} \begin{vmatrix} -1 & 2 \\ 5 & -2 \end{vmatrix}$$

$$= \frac{-8}{84} = -\frac{2}{21}.$$