Practice Problems S4 with Solutions

Note: Please review sections 3.3 and 3.4 for quiz #4. Complex numbers will be on final Quiz.

1. Compute $P^{-1}AP$ and then A^n if $A = \begin{bmatrix} 6 & -5 \\ 2 & -1 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 5 \\ 1 & 2 \end{bmatrix}$

2. (Diagonalization) Find the characteristic polynomial , eigenvalues and an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix if $A = \begin{bmatrix} 3 & 1 & 1 \\ -4 & -2 & -5 \\ 2 & 2 & 5 \end{bmatrix}$.

3. Determine whether the following matrices are diagonalizable or not: (a) $A = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$; (b) $B = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$.

4. Consider a linear dynamical system $V_{k+1} = A\,V_k$ for $k \geq 0$. Using the dominant eigenvalue of A, approximate V_k if:

   (a) $A = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}$, $V_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$; (b) $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 3 \\ 1 & 4 & 1 \end{bmatrix}$, $V_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

5. Solve the linear recurrence $x_{k+2} = 3x_k + 2x_{k+1}$, where $x_0 = 1 = x_1$. Find x_8.

6. Approximate the k^{th} number x_k of a sequence given by the linear recurrence $x_{k+3} = 6x_{k+2} - 11x_{k+1} + 6x_k$, where $x_0 = 1 = x_2$ and $x_1 = 0$.

Recommended Problems:

Pages 141: 1 a, b, c, d, e, h; 2 b, d; 3, 4, 8 d; 10, 13, 14;
Page 147: 1 b, c, d; 2 b

Solutions

1. $P^{-1} = \begin{bmatrix} 1 & 5 \\ 1 & 2 \end{bmatrix}^{-1} = -\frac{1}{3}\begin{bmatrix} 2 & -5 \\ -1 & 1 \end{bmatrix}$. So, $P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} =$ diag$(1,4)$. It follows that $A = P\text{diag}(1,4)P^{-1}$. Therefore,

$$\begin{aligned} A^n &= P\text{diag}(1,4^n)P^{-1} \\ &= -\frac{1}{3}\begin{bmatrix} 1 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4^n \end{bmatrix}\begin{bmatrix} 2 & -5 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -2/3 + 5/3\,4^n & 5/3 - 5/3\,4^n \\ -2/3 + 2/3\,4^n & 5/3 - 2/3\,4^n \end{bmatrix}. \end{aligned}$$

2. The characteristic polynomial of A is

$$\det(xI_3 - A) = \begin{vmatrix} x-3 & -1 & -1 \\ 4 & x+2 & 5 \\ -2 & -2 & x-5 \end{vmatrix} = (x-1)(x-2)(x-3).$$

Thus A has three eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 3$. Eigenvectors: The homogeneous systems $(3I_3 - A)X = 0$, $(2I_3 - A)X = 0$ and $(I_3 - A)X = 0$ have basic solutions $X_1 = [1\ {-3}\ 1]^T$, $X_2 = [1\ {-1}\ 0]^T$ and $X_3 = [0\ -1\,1\,]^T$. There are basic eigenvectors correponding to the respective eigenvalues. The matrix $P = [X_1\,X_2\,X_3] = \begin{bmatrix} 1 & 1 & 0 \\ -3 & -1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$ diagonalizes A with $P^{-1}AP = \text{diag}(1,2,3)$.

3. Since the 2×2 matrix A has two distinct eigenvalues $\lambda_1 = -5$ and $\lambda_2 = 2$, A is diagonalizable. The matrix B has an eigenvalue $\lambda = 1$ with multiplicity 2. For B to be diagonalizable, there must be two basic eigenvectors corresponding to $\lambda = 1$.But the homogeneous system $(I_3 - B)X$ has only one basic solution $X = [0\ 1\ 0]^T$. Therefore, B is not diagonalizable.

4. (a) $V_k = A^kV_0$, where the matrix A has eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 3$ with respective basic eigenvectors $X_1 = [1\ -\ 4]^T$ and $X_1 = [1\ 1]^T$. The matrix $P = [X_1\ X_2]$ diagonalizes A. $B = P^{-1}V_0 = \frac{1}{5}[-1\ 6]^T$, i.e. $b_1 = -\frac{1}{5}$ and $b_2 = \frac{6}{5}$. We have: $V_k =$

$b_1{\lambda_1}^k X_1 + b_2{\lambda_2}^k X_2 = -\frac{1}{5}(-2)^k[1\ -4]^T + \frac{6}{5}3^k[1\ \ 1]^T$ is the exact k^{th} vector V_k. Since $\lambda_2 = 3$, is the dominant eigenvalue of A, $V_k \approx \frac{6}{5}3^k[1\ \ 1]^T$.

(b) With $P = \begin{bmatrix} 0 & -4 & 0 \\ 1 & 1 & 3 \\ 1 & 1 & -4 \end{bmatrix}$, we have $P^{-1}AP = \text{diag}(5, -2, 1)$, i.e, $X_3 = [0\ 3\ -4]^T$, $X_2 = [-4\ 1\ 1]^T$ and $X_1 = [0\ 1\ 1]^T$ are eigenvectors corresponding to the eigenvalues $\lambda_3 = -2$, $\lambda_2 = 1$ and $\lambda_3 = 5$, respectively. $P^{-1}V_0 = \frac{5}{4}[1\ -1\ 1] = [b_1\ b_2\ b_3]$ and $\lambda_1 = 5$ is the dominant eigenvalue of A, so $V_k \approx b_1\lambda_1^k X_1 = \frac{5}{4}5^k[0\ 1\ 1]^T$.

5. Define $V_k = [x_k\ x_{k+1}]^T$. Then $V_0 = [x_0\ x_1] = [1\ 1]^T$ and $V_{k+1} = [x_{k+1}\ x_{k+2}]^T = [x_{k+1}\ 3x_k + 2x_{k+1}]^T] = AV_k$. It follows that $V_k = A^k V_0$, where $A = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}$. Diagonalizing A, we have $P^{-1}AP = \text{diag}(3, -1)$, with diagonalizing matrix $P = [X_1\ X_2] = \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix}$. $B = P^{-1}V_0 = \frac{1}{2}[1\ -1]^T$. So, $V_k = b_1\lambda_1^k X_1 + b_2\lambda_2^k X_2 = \frac{1}{2}3^k[1\ 3]^T - \frac{1}{2}(-1)^k[-1\ 1] = [\frac{3^k}{2} + \frac{(-1)^k}{2}\ \ \frac{3\times 3^k}{2} - \frac{(-1)^k}{2}]^T = [x_k\ x_{k+1}]^T$. Thus, $x_k = \frac{3^k}{2} + \frac{(-1)^k}{2}$. $x_8 = \frac{3^8}{2} + \frac{(-1)^8}{2} = \frac{3^8}{2} + \frac{1}{2} = 3281$.

6. With $V_k = [x_k\ x_{k+1}\ x_{k+2}]^T$, we have $V_0 = [1\ 0\ 1]^T$ and $V_{k+1} = AV_k$; so, $V_k = A^k V_0$, where the matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix}$ is diagonalizable with eigenvalues $\lambda_1 = 3$, $\lambda_2 = 2$ and $\lambda_3 = 1$ and basic eigenvectors given by $X_1 = [1\ 3\ 9]^T$, $X_2 = [1\ 2\ 4]^T$ and $X_3 = [1\ 1\ 1]^T$. A is diagonalized by $P = [X_1\, X_2\, X_3]$ with inverse $P^{-1} = \begin{bmatrix} 1 & -3/2 & 1/2 \\ -3 & 4 & -1 \\ 3 & -5/2 & 1/2 \end{bmatrix}$. So, $B = [b_1\ b_2\ b_3]^T = P^{-1}V_0 = [\frac{3}{2}\ \ -4\ \ \frac{7}{2}]^T$. We have, $V_k = b_1\lambda_1^k X_1 + b_2\lambda_2^k X_2 + b_3\lambda_3^k X_3 \approx b_1\lambda_1^k X_1 = \frac{3}{2}3^k[1\ 3\ 9]^T$. Finally for k very large, $x_k \approx \frac{3^{k+1}}{2}$.

(The exact value is $x_k = \frac{7}{2} + \frac{3^{k+1}}{2} - 2^{k+2}$)