Practice Problems S5 (Complex Numbers)

1. Write the following complex numbers in the form $a + bi$:

   (a) $\frac{3-i}{2i+5}$, (b) $(2-3i)^3$, (c) $\frac{1-i}{2-3i} - \frac{1+2i}{5+i}$, (d) $e^{5i\pi/3}$.

2. Express the following complex numbers in polar from:

   (a) $(1-\sqrt{3}i)^5$, (b) $(\sqrt{3}-i)(2-2i)$.

3. (a) Express the number $z = (1+i)(1+\sqrt{3}i)$ in polar form and in the form $a + bi$;

   (b) Find $\cos(7\pi/12)$ and $\sin(7\pi/12)$.

4. Solve the following equations:

   (a) $(i+z) - 3i(2-z) = iz + 1$;

   (b) $z(1+i) = \bar{z} - (3+2i)$;

   (c) $3x^2 + 5x + 10 = 0$;

   (d) $z^2 - (3-2i)z + (5-i) = 0$.

5. Solve the following system of linear equations:

$$\left\{ \begin{array}{r} x + \ iy - iz = 3 + i \\ -ix + \ 2y + iz = 2 \\ (i-1)x - (1+2i)y + 2z = i - 1 \end{array} \right. .$$

6. Find the inverse of $A = \begin{bmatrix} 1 & 1-i \\ 2+i & 3+i \end{bmatrix}$.

7. Diagonalize the matrix $A = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$.

8. Find the fifth roots of $w = 16(-\sqrt{3} + i)$.

Recommended Problems:

Pages 482 - 483: 1, 2, 3 a, 4 a, b; 5 a, b, c; 6 a, b, d; 10 a, b;11 a, b, c; 18 , 19, 23.

Solutions

1. (a) $\frac{3-i}{2i+5} = \dfrac{(3-i)(5-2i)}{|2i+5|^2} = \dfrac{13-11i}{29} = \dfrac{13}{29} - \dfrac{11}{29}i$; (b) $(2-3i)^3 = -46-9i$; (c) $e^{5\pi/3} = \cos(5\pi/3)) + \sin(5\pi/3)i = \dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i$.

2. (a) $(1-\sqrt{3}i)^5 = \left(2(1/2 - \sqrt{3}/2i\right)^5 = \left(2e^{-\pi i/3}\right)^5 = 32e^{-5\pi i/3} = \dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$; (b) $(\sqrt{3} - i)(2-2i) = 2e^{-\pi i/6}(2\sqrt{2}e^{-\pi i/4}) = 4\sqrt{2}e^{-i\pi/4 - i\pi/6} = 4\sqrt{2}e^{-5\pi i/12}$.

3. (a) $z = (1+i)(1+\sqrt{3}i) = 2\sqrt{2}e^{\pi i/4}e^{\pi i/3} = 2\sqrt{2}e^{\pi i/4 + \pi i/3} = 2\sqrt{2}e^{7\pi i/12}$; (b) Since $2\sqrt{2}e^{7\pi i/12} = z = (1+i)(1+\sqrt{3}i) = (1-\sqrt{3}) + (1-\sqrt{3})i$, we have $\cos(7\pi i/12) = \dfrac{\sqrt{2}(1-\sqrt{3})}{4}$ and $\cos(7\pi i/12) = \dfrac{\sqrt{2}(1+\sqrt{3})}{4}$.

4. (a) $z = 11/5 + 3i/5$; (b) $z = 2 + 3i$, (c) $x = -5/6 - i\sqrt{95}/6$ or $x = -5/6 + i\sqrt{95}/6$. (d) $z = 2 - 3i$ or $z = 1 + i$.

5. Carry the augmented matrix to reduced row-echelon form:

$$\begin{bmatrix} 1 & 0 & 1-2\,i & 6 \\ 0 & 1 & 1+i & 1+3\,i \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which gives $x = 6 - (1-2i)t$, $y = 1 + 3i - (1+i)t$ and $z = t$, where $t \in \mathbb{C}$.

6. A has inverse $\begin{bmatrix} 1/2 - 3/2\,i & 1/2 + 1/2\,i \\ -1/2 + i & -1/2\,i \end{bmatrix}$.

7. A has eigenvalues $\lambda_1 = 1 - i$ and $\lambda_1 = 1 + i$ and corresponding basic eigenvectors $X_1 = [-1 \;\; 1]^T$ and $X_2 = [1 \;\; 1]^T$. Therefore, the matrix $P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$, i.e., $P^{-1}AP = \text{diag}(1 - i, 1 + i)$.