

Practice Problems S6 (Vector Geometry)

- Let $P_1(2, 1, -2)$ and $P_2(1, -2, 0)$ be points in \mathbb{R}^3 .
 - Find the parametric equations of the line through P_1 and P_2 ;
 - Find the coordinates of the point P that is $1/4$ of the way from P_1 to P_2 .

- Find the point of intersection P between the lines (if they are concurrent):

$$\begin{cases} x = 3 + t \\ y = 2 + 3t \\ z = -1 - 3t \end{cases} \quad \text{and} \quad \begin{cases} x = 1 - s \\ y = 1 + 2s \\ z = 3 + s \end{cases} .$$

- Find the equation of the plane passing through the point $P(3, -7, 5)$ and is perpendicular to the line $\begin{cases} x = 2 + 6t \\ y = -5 - 6t \\ z = 3 + 5t \end{cases}$

- Find the equation of the plane through the points $A(3, -7, 1)$, $B(2, 0, -1)$ and $C(1, 3, 0)$. Check if the point $D(5, 1, 1)$ lies on this plane.

- Determine whether the plane $2x - 3y + z = 1$ contains the line $\begin{cases} x = 3 + 2t \\ y = 2 \\ z = 1 - 4t \end{cases}$.

- Find the line of intersection of the planes $(\pi_1) \equiv 3x + 5y + 4z = 5$ and $(\pi_2) \equiv x + 2y + 3z = 2$.

- find the shortest distance from the point $P(1, 1, 1)$ to the line $\begin{cases} x = 3 + t \\ y = 2 \\ z = 1 - 4t \end{cases}$.

Which point on this line is closest to P .

8. the shortest distance from the point $P(1, 1, 1)$ to the plane $x - 4z = 2$. Which point on this plane is closest to P .
9. Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear transformation. Find the matrix that induces T if:
 - (a) T is the projection on the line through the origin with direction vector $\vec{d} = [1 \ 3 \ -3]^T$.
 - (b) T is the reflection about (l)
10. Find the areas of the sides of the parallelepiped determined by the vectors \vec{AB} , \vec{AC} , and \vec{AD} , where A, B, C and D are the points in Problem 4. What is the volume of this parallelepiped?

Recommended Problems:

Pages 165 - 167: 1 a, c; 3 a; 4 a; 5, 7 a; 9 a, b; 15, 20, 24

Pages 177 - 179: 1 a; 2 a, b; 3 a; 6, 8, 9, 10 a; 11 a; 12, 13, 14, 15, 16 a; 18, 19, 24 a

Page 185: 3 a; 4 a; 5 a;

Solutions

1. (a) If $P(x, y, z)$ is a point on the line through P_1 and P_2 , then there is $t \in \mathbb{R}$ such that $\overrightarrow{P_1P} = t\overrightarrow{P_1P_2} = t[-1 \ -3 \ 2]$. Thus,
$$\begin{cases} x = 2 - t \\ y = 1 - 3t \\ z = -2 + 2t \end{cases}.$$

(b) $\overrightarrow{P_1P} = \frac{1}{4}\overrightarrow{P_1P_2}$. This gives $P(7/4, 1/4, -3/2)$.

2. Solve the following system of linear equations
$$\begin{cases} x = 3 + t = 1 - s \\ y = 2 + 3t = 1 + 2s \\ z = -1 - 3t = 3 + s \end{cases}$$
 to get $t = -1 = s$. The point of intersection P has coordinates $P(2, -1, 2)$.

3. The plane has normal vector $\vec{v} = [6 \ -6 \ 5]^T$ (direction vector of the given line). So, the scalar equation is $6(x - 3) - 6(y + 7) + 5(z - 5) = 0$.
4. If $P(x, y, z)$ is a point on the plane through A , B and C , then the equation of the plane is

$$\begin{aligned} 0 &= \overrightarrow{AP} \cdot (\overrightarrow{AB} \times \overrightarrow{AC}) = \det(\overrightarrow{AP}, \overrightarrow{AB}, \overrightarrow{AC}) = \begin{vmatrix} x - 3 & y + 7 & z - 1 \\ 2 - 3 & 0 + 7 & -1 - 1 \\ 1 - 3 & 3 + 7 & 0 - 1 \end{vmatrix} \\ &= 13(x - 3) + 3(y + 7) + 4(z - 1). \end{aligned}$$

Replacing x , y and z by the coordinates of $D(5, 1, 1)$, we have: $13(5 - 3) + 3(1 + 7) + 4(1 - 1) = 26 + 24 = 50 \neq 0$. Therefore this plane does not contain the point D .

5. A line can be completely on the given plane, parallel to the plane or can intersect the plane at one point. Replace the parametric expressions of x , y and z into the equation of the plane. If one gets a consistent equation in the parameter t , then the line intersects the plane. If the parameter t is cancelled out, then the line is contained in the plane if the equation is consistent, otherwise, the line is parallel to the plane. With $x = 3 + 2t$, $y = 2$ and $z = 1 - 4t$, we have: $2(3 + 2t) - 3(2) + (1 - 4t) = 1$, i.e. $1 = 1$, this implies that the plane contains the line.

6. Solve $\begin{cases} 3x + 5y + 4z = 5 \\ x + 2y + 3z = 2 \end{cases}$ to find the line of intersection $\begin{cases} x = 7t \\ y = 1 - 5t \\ z = t \end{cases}$.

7. The line has direction vector $\vec{d} = [1 \ 0 \ -4]^T$. Choose arbitrarily a point $P_0 = (3, 2, 1)$ on the line. Project the vector $\overrightarrow{P_0P} = [-2 \ -1 \ 0]^T$ on \vec{d} : $\text{proj}_{\vec{d}}\overrightarrow{P_0P} = \frac{\overrightarrow{P_0P} \cdot \vec{d}}{\|\vec{d}\|^2} \vec{d} = -\frac{2}{17} \vec{d}$. The shortest distance is $\|\overrightarrow{P_0P} - \text{proj}_{\vec{d}}\overrightarrow{P_0P}\| = \|\overrightarrow{P_0P} - \frac{2}{17} \vec{d}\| = \frac{1}{17} \sqrt{32^2 + 17^2 + 8^2}$. The closest point $Q(x, y, z)$ is given by $\overrightarrow{P_0Q} = \text{proj}_{\vec{d}}\overrightarrow{P_0P}$. So, $Q(49/17, 2, 25/17)$.

8. The plane has normal vector $\vec{n} = [1 \ 0 \ -4]^T$. Choose arbitrarily a point $P_0(6, 1, 1)$ on the plane. Project the vector $\overrightarrow{P_0P} = [-5 \ 0 \ 0]^T$ on \vec{n} : The shortest distance is given by $\|\text{proj}_{\vec{n}}\overrightarrow{P_0P}\| = \|\frac{5}{17} \vec{n}\| = \frac{5}{17} \sqrt{17}$. The closest point $Q(x, y, z)$ is given by $\overrightarrow{QP} = \text{proj}_{\vec{n}}\overrightarrow{QP} = -\frac{5}{17} \vec{n}$. So, $Q(22/17, 1, -3/17)$.

9. (a) If $T = P_{(l)}$ is the projection of vectors $\vec{v} = [x \ y \ z]^T$ on the line (l) with direction vector $\vec{d} = [1 \ 3 \ -3]^T$, then

$$\begin{aligned} T(\vec{v}) &= P_{(l)}(\vec{v}) = \text{proj}_{\vec{d}}\vec{v} = \frac{\vec{v} \cdot \vec{d}}{\|\vec{d}\|^2} \vec{d} = \frac{x + 3y - 3z}{1 + 9 + 9} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ &= \frac{1}{19} \begin{bmatrix} x + 3y - 3z \\ 3x + 9y - 9z \\ -3x - 9y + 9z \end{bmatrix} = \frac{1}{19} \begin{bmatrix} 1 & 3 & -3 \\ 3 & 9 & -9 \\ -3 & -9 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \end{aligned}$$

Therefore, T is a matrix transformation induced by the matrix $A_{P_{(l)}} =$

$$\frac{1}{19} \begin{bmatrix} 1 & 3 & -3 \\ 3 & 9 & -9 \\ -3 & -9 & 9 \end{bmatrix}.$$

(b) If $T = Q_{(l)}$ is the reflection of vectors $\vec{v} = [x \ y \ z]^T$ about the line (l) , then $T(\vec{v}) = Q_{(l)}(\vec{v}) = 2\text{proj}_{\vec{d}}\vec{v} - \vec{v} = A_{Q_{(l)}}(\vec{v})$, where $A_{Q_{(l)}} = 2A_{P_{(l)}} - I_3$.

10. The parallelepiped has six sides (2 times the parallelograms determined by the vectors). So, these sides have areas: $\|\vec{AB} \times \vec{AC}\| = \|[13 \ 3 \ 4]^T\| = \sqrt{194}$, $\|\vec{AB} \times \vec{AD}\| = \|[16 \ -4 \ -22]^T\| = \sqrt{756}$ and $\|\vec{AC} \times \vec{AD}\| = \|[8 \ -2 \ -36]^T\| = 2\sqrt{341}$. The volume of this parallelepiped is $Vol = |\vec{AD} \cdot (\vec{AB} \times \vec{AC})| = |\det(\vec{AD}, \vec{AB}, \vec{AC})| = 50$.