Practice Problems S4

1. By inspection, find the determinants of the following matrices:
(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$; (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; (c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$;
(d) $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 4 \\ 2 & -4 & 6 \end{bmatrix}$; (e) $\begin{bmatrix} 1 & 0 & 4 & 9 \\ -8 & -7 & 12 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 5 & 3 \end{bmatrix}$.

2. Compute the determinants of the following matrices
(a) $A = \begin{bmatrix} -2 & 1 & 3 \\ 1 & -7 & 4 \\ -2 & 1 & 3 \end{bmatrix}$; (b) $A = \begin{bmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{bmatrix}$.

3. Find the inverse of $A = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$ using the adjoint formula.

4. Given $A = \begin{bmatrix} 3 & -1 & 2 \\ 5 & 5 & -2 \\ 1 & 2 & 3 \end{bmatrix}$, find the $(1, 3)$-entry of A^{-1}.

5. For which values of $c \in \mathbb{R}$ is A invertible if $A = \begin{bmatrix} 1 & c & 0 \\ 2 & 0 & c \\ c & -1 & 1 \end{bmatrix}$.

6. Solve the following system by Cramer's rule:
(a) $\begin{cases} x + 2y = 4 \\ 3x + 7y = 13 \end{cases}$; (b) $\begin{cases} 3x - 2y + 4z = -3 \\ 5x + 3y + z = 0 \\ 2x + 6y - 5z = 6 \end{cases}$.

Solutions

1. (a): -1; (b): -3; (c)1; (d): 0; (e):-21

2. (a): 0; (b): $\det(A) = \begin{vmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{vmatrix} = -\begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 1 & 8 & 0 \end{vmatrix} =$
$-\begin{vmatrix} 0 & 9 & 3 \\ 0 & 3 & 3 \\ 1 & 8 & 0 \end{vmatrix} = -\begin{vmatrix} 9 & 3 \\ 3 & 3 \end{vmatrix} = -18.$

3. $A^{-1} = \dfrac{1}{\det(A)}\text{adj(A)}$, where $\det(A) = -1$ and $\text{adj(A)} = [c_{ij}(A)]^T =$
$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & -1 \\ -1 & 1 & -1 \end{bmatrix}^T = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}$. Therefore, $A^{-1} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$.

4. A has determinat $\det(A) = 84$.The $(\mathbf{1}, \mathbf{3})$-entry of A^{-1} is $\frac{1}{\det(A)} c_{\mathbf{31}}(A) =$
$\frac{1}{84}(-1)^{3+1} \begin{vmatrix} -1 & 2 \\ 5 & -2 \end{vmatrix} = -\frac{8}{84} = -\frac{2}{21}$.

5. A is invertible iff $0 \neq \det(A) = c^3 - c = c(c-1)(c+1)$. So, A is invertible iff $c \in \mathbb{R} \setminus \{1, 0, -1\}$.

6. (a) $x = \dfrac{\begin{vmatrix} 4 & 2 \\ 13 & 7 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 3 & 7 \end{vmatrix}} = \dfrac{28 - 26}{7 - 6} = 2$; $y = \dfrac{\begin{vmatrix} 1 & 4 \\ 3 & 13 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 3 & 7 \end{vmatrix}} = \dfrac{13 - 12}{7 - 6} = 1.$

(b) $x = \dfrac{\begin{vmatrix} -3 & -2 & 4 \\ 0 & 3 & 1 \\ 6 & 6 & -5 \end{vmatrix}}{\begin{vmatrix} 3 & -2 & 4 \\ 5 & 3 & 1 \\ 2 & 6 & -5 \end{vmatrix}} = \dfrac{-21}{-21} = 1$; $y = \dfrac{\begin{vmatrix} 3 & -3 & 4 \\ 5 & 0 & 1 \\ 2 & 6 & -5 \end{vmatrix}}{\begin{vmatrix} 3 & -2 & 4 \\ 5 & 3 & 1 \\ 2 & 6 & -5 \end{vmatrix}} = \dfrac{21}{-21} =$

$$-1;\ z=\frac{\begin{vmatrix} 3 & -2 & -3 \\ 5 & 3 & 0 \\ 2 & 6 & 6 \end{vmatrix}}{\begin{vmatrix} 3 & -2 & 4 \\ 5 & 3 & 1 \\ 2 & 6 & -5 \end{vmatrix}}=\frac{42}{-21}=-2.$$