Practice Problems S5 (Diagonalization, Linear Recurrences)

1. Compute $P^{-1}AP$ and then A^n if $A = \begin{bmatrix} 6 & -5 \\ 2 & -1 \end{bmatrix}$ and $P = \begin{bmatrix} 5 & 1 \\ 2 & 1 \end{bmatrix}$

2. (Diagonalization) Find the characteristic polynomial, eigenvalues and an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix if $A = \begin{bmatrix} 3 & 1 & 1 \\ -4 & -2 & -5 \\ 2 & 2 & 5 \end{bmatrix}$.

3. Determine whether the following matrices are diagonalizable or not: (a) $A = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$; (b) $B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$; (c) $C = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$.

4. Solve the following linear recurrences:

   (a) $x_{k+2} = 2x_k - x_{k+1}$, where $x_0 = 1$ and $x_1 = 2$;

   (b) $x_{k+3} = -2x_k + x_{k+1} + 2x_{k+2}$, where $x_0 = 1$ and $x_1 = 2 = x_2$.

Solutions

1. $P^{-1} = \begin{bmatrix} 5 & 1 \\ 2 & 1 \end{bmatrix}^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ -2 & 5 \end{bmatrix}$. So, $P^{-1}AP = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} =$ diag$(4,1)$. It follows that $A = P\text{diag}(4,1)P^{-1}$. Therefore,

$$\begin{aligned} A^n &= P\text{diag}(4^n,1)P^{-1} \\ &= \frac{1}{3}\begin{bmatrix} 5 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 4^n & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -2 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 5/3\,4^n - 2/3 & -5/3\,4^n + 5/3 \\ 2/3\,4^n - 2/3 & -2/3\,4^n + 5/3 \end{bmatrix}. \end{aligned}$$

2. The characteristic polynomial of A is

$$\det(xI_3 - A) = \begin{vmatrix} x-3 & -1 & -1 \\ 4 & x+2 & 5 \\ -2 & -2 & x-5 \end{vmatrix} = (x-1)(x-2)(x-3).$$

Thus A has three eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 3$. Eigenvectors: The homogeneous systems $(I_3 - A)X = 0$, $(2I_3 - A)X = 0$ and $(3I_3 - A)X = 0$ have basic solutions $X_1 = [1\ -3\ 1]^T$, $X_2 = [1\ -1\ 0]^T$ and $X_3 = [0\ -1\ 1]^T$, respectively. There are basic eigenvectors correponding to the respective eigenvalues. The matrix $P = [X_1\, X_2\, X_3] = \begin{bmatrix} 1 & 1 & 0 \\ -3 & -1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$ diagonalizes A with $P^{-1}AP = \text{diag}(1,2,3)$.

3. (a) Since the 2×2 matrix A has two distinct (or simple, i.e., with multiplicity 1) eigenvalues $\lambda_1 = -5$ and $\lambda_2 = 2$, A is diagonalizable.

   (b) The matrix B has eigenvalue $\lambda = -1$ with **multiplicity** 2. The matrix B is diagonalizable if there are two basic eigenvectors corresponding $\lambda = -1$. There are basic solutions to the homogeneous system $(-I_3 - B)X = 0$. So, $X_1 = [-1\ 1, 0]$ and $X_2 = [-1, 0, 1]$ are **two basic eigenvectors** corresponding to $\lambda = -1$. The matrix B is diagolizable.

   (c) The matrix C has an eigenvalue $\lambda = 1$ of multiplicity 2. For C to be diagonalizable, there must be two basic eigenvectors corresponding to $\lambda = 1$. But the homogeneous system $(I_3 - C)X = 0$

has only one basic solution $X = [0\ 1\ 0]^T$. Therefore, C is not diagonalizable.

4. (a) Define $V_k = \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}$ for all $k \geq 0$. We have $V_0 = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $V_{k+1} = \begin{bmatrix} x_{k+1} \\ x_{k+2} \end{bmatrix} = \begin{bmatrix} x_{k+1} \\ 2x_k - x_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} V_k$. We obtain a linear dynamical system: $V_k = A^k V_0$.

Diagonalization of $A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$: $c_A(x) = \det(xI_2 - A) = \begin{vmatrix} x & -1 \\ -2 & x+1 \end{vmatrix} = x(x+1) - 2 = x^2 + x - 2 = (x-1)(x+2)$. A has two simple eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 1$. Solving the homogeneous systems $(-2I_2 - A)X = 0$ and $(I_2 - A)X = 0$, we find basic eigenvectors $X_1 = [1\ -2]^T$ and $X_2 = [1\ 1]^T$ corresponding to the eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 1$, respectively. So, $P = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$, with inverse $P^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$ diagonalizes A. $B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = P^{-1}V_0 = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -1 \\ 4 \end{bmatrix}$. It follows that $\begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} = V_k = b_1\lambda_1^k X_1 + b_2\lambda_2^k X_2 = \frac{-1}{3}(-2)^k X_1 + \frac{4}{3}X_2 = \frac{1}{3}\begin{bmatrix} -(-2)^k + 4 \\ 2(-2)^k + 4 \end{bmatrix}$. Therefore, $x_k = \left(4 - (-2)^k\right)/3$.

(b) Define $V_k = \begin{bmatrix} x_k \\ x_{k+1} \\ x_{k+2} \end{bmatrix}$ for all $k \geq 0$. We have $V_0 = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ and

$$\begin{aligned} V_{k+1} &= \begin{bmatrix} x_{k+1} \\ x_{k+2} \\ x_{k+3} \end{bmatrix} = \begin{bmatrix} x_{k+1} \\ x_{k+2} \\ -2x_k + x_{k+1} + 2x_{k+2} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_k \\ x_{k+1} \\ x_{k+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{bmatrix} V_k. \end{aligned}$$

The matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{bmatrix}$ has eigenvalues $\lambda_1 = 2$, $\lambda_2 = -1$ and $\lambda_3 = 1$ to which correspond the basic eigenvectors $X_1 = [1\ 2\ 4]^T$, $X_2 = [1\ -1\ 1]^T$ and $X_3 = [1\ 1\ 1]^T$, respectively. So, $P = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 4 & 1 & 1 \end{bmatrix}$ with inverse $P^{-1} = \frac{1}{6}\begin{bmatrix} -2 & 0 & 2 \\ 2 & -3 & 1 \\ 6 & 3 & -3 \end{bmatrix}$, is a diagonalizing matrix for A. With $B = [b_1\ b_2\ b_3]^T = P^{-1}V_0 = [1/3, -1/3, 1]$, we have $V_k = b_1\lambda_1^k X_1 + b_2\lambda_2^k X_2 + b_3\lambda_3^k X_3 = 2^k/3X_1 - (-1)^k/3X_2 + X_3 = \begin{bmatrix} 2^k/3 - (-1)^k/3 + 1 \\ 22^k/3 + (-1)^k/3 + 1 \\ 42^k/3 - (-1)^k/3 + 1 \end{bmatrix}$. Therefore, $x_k = 2^k/3 - (-1)^k/3 + 1$.