

## Practice Problems S6 (Complex Numbers)

- Write the following complex numbers in the form  $a + bi$ :  
(a)  $\frac{3-i}{2i+5}$ , (b)  $(2 - 3i)^3$ , (c)  $\frac{1-i}{2-3i} - \frac{1+2i}{5+i}$ , (d)  $e^{5i\pi/3}$ .
- Express the following complex numbers in polar form: (a)  $(1 - \sqrt{3}i)^5$ ,  
(b)  $(\sqrt{3} - i)(2 - 2i)$ , (c)  $-2e^{\pi i/3}$
- Prove that  $\cos(\theta_1 + \theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)$  and  $\sin(\theta_1 + \theta_2) = \cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2)$ .
- (a) Express the number  $z = (1 - i)(-1 + \sqrt{3}i)$  in polar form and in the form  $a + bi$ ;  
(b) Find  $\cos(5\pi/12)$  and  $\sin(5\pi/12)$ .
- Solve the following equations:  
(a)  $(i + z) - 3i(2 - z) = iz + 1$ ;  
(b)  $z(1 + i) = \bar{z} - (3 + 2i)$ ;  
(c)  $3x^2 + 5x + 10 = 0$ ;  
(d)  $z^2 = -15 - 8i$ ;  
(e)  $z^2 - (3 - 2i)z + (5 - i) = 0$ .
- Solve the following system of linear equations:

$$\begin{cases} x + iy - iz & = 3 + i \\ -ix + 2y + iz & = 2 \\ (i - 1)x - (1 + 2i)y + 2z & = i - 1 \end{cases}.$$

- Find the inverse of  $A = \begin{bmatrix} 1 & 1 - i \\ 2 + i & 3 + i \end{bmatrix}$ .

8. Diagonalize the matrix  $A = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$ .

9. Find the 8th roots of  $z = 128(-1 - \sqrt{3}i)$ .

**Recommended Problems:**

Pages 482 - 483: 1, 2, 3 a, 4 a, b; 5 a, b, c; 6 a, b, d; 10 a, b; 11 a, b, c; 18 ,  
19, 23.

### Solutions

1. (a)  $\frac{3-i}{2i+5} = \frac{(3-i)(5-2i)}{|2i+5|^2} = \frac{13-11i}{29} = \frac{13}{29} - \frac{11}{29}i$ ; (b)  $(2-3i)^3 = -46-9i$ ; (c)  $e^{5\pi/3} = \cos(5\pi/3) + \sin(5\pi/3)i = \frac{1}{2} - \frac{\sqrt{3}}{2}i$ .

2. (a)  $(1-\sqrt{3}i)^5 = (2(1/2-\sqrt{3}/2i))^5 = (2e^{-\pi i/3})^5 = 32e^{-5\pi i/3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ ; (b)  $(\sqrt{3}-i)(2-2i) = 2e^{-\pi i/6}(2\sqrt{2}e^{-\pi i/4}) = 4\sqrt{2}e^{-i\pi/4-i\pi/6} = 4\sqrt{2}e^{-5\pi i/12}$ ; (c)  $-2e^{\pi i/3} = 2(-1)e^{\pi i/3} = 2e^{\pi i}e^{\pi i/3} = 2e^{(\pi+\pi/3)i} = 2e^{4\pi i/3}$ .

3.  $e^{i\theta_1}e^{i\theta_2} = e^{(\theta_1+\theta_2)i}$  (Multiplication rule). It follows that

$$\begin{aligned} (\cos(\theta_1) + i \sin(\theta_1)) (\cos(\theta_2) + i \sin(\theta_2)) &= \cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2) \\ \cos(\theta_1) \cos(\theta_2) - \sin(\theta_1) \sin(\theta_2) + i (\cos(\theta_1) \sin(\theta_2) + \sin(\theta_1) \cos(\theta_2)) & \\ &= \cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2). \end{aligned}$$

Therefore,  $\cos(\theta_1 + \theta_2) = \cos(\theta_1) \cos(\theta_2) - \sin(\theta_1) \sin(\theta_2)$  and  $\sin(\theta_1 + \theta_2) = \cos(\theta_1) \sin(\theta_2) + \sin(\theta_1) \cos(\theta_2)$ .

4. (a)  $z = (1-i)(-1+\sqrt{3}i) = 2\sqrt{2}e^{-\pi i/4}e^{2\pi i/3} = 2\sqrt{2}e^{-\pi i/4+2\pi i/3} = 2\sqrt{2}e^{5\pi i/12}$ ; (b) Since  $2\sqrt{2}e^{5\pi i/12} = z = (1-i)(-1+\sqrt{3}i) = (-1+\sqrt{3}) + (1+\sqrt{3})i$ , we have  $\cos(5\pi i/12) = \frac{\sqrt{2}(\sqrt{3}-1)}{4}$  and  $\sin(5\pi i/12) = \frac{\sqrt{2}(1+\sqrt{3})}{4}$ .

5. (a)  $z = 11/5 + 3i/5$ ; (b)  $z = 2 + 3i$ ; (c)  $x = -5/6 - i\sqrt{95}/6$  or  $x = -5/6 + i\sqrt{95}/6$ ; (d)  $z = 1 - 4i$  or  $z = -1 + 4i$ ; (e)  $z = 2 - 3i$  or  $z = 1 + i$ .

6. Carry the augmented matrix to reduced row-echelon form:

$$\begin{bmatrix} 1 & 0 & 1-2i & 6 \\ 0 & 1 & 1+i & 1+3i \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which gives  $x = 6 - (1-2i)t$ ,  $y = 1 + 3i - (1+i)t$  and  $z = t$ , where  $t \in \mathbb{C}$ .

7.  $A$  has inverse  $\begin{bmatrix} 1/2 - 3/2i & 1/2 + 1/2i \\ -1/2 + i & -1/2i \end{bmatrix}$ .

8.  $A$  has eigenvalues  $\lambda_1 = 1 - i$  and  $\lambda_2 = 1 + i$  and corresponding basic eigenvectors  $X_1 = [-1 \ 1]^T$  and  $X_2 = [1 \ 1]^T$ . Therefore, the matrix  $P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ , i.e.,  $P^{-1}AP = \text{diag}(1 - i, 1 + i)$ .

9. The complex number  $z = 128(-1 - \sqrt{3}i) = 256e^{4\pi i/3} = 2^8e^{4\pi i/3}$  has 8th roots given by  $z_k = 256^{1/8}e^{\frac{4\pi/3 + 2k\pi}{8}i} = 2e^{(\pi/6 + k\pi/4)i}$ ,  $k = 0, 1, 2, 3, \dots$

So,  $z_0 = 2e^{\pi/6i} = 2(\cos(\pi/6) + i\sin(\pi/6)) = \sqrt{3} + i$ ;

$$z_1 = 2e^{(\pi/6 + \pi/4)i} = 2e^{5\pi i/12} = \frac{\sqrt{2}}{2}(\sqrt{3} - 1 + i(1 + \sqrt{3}));$$

$$z_2 = 2e^{(\pi/6 + 2\pi/4)i} = 2e^{2\pi i/3} = -1 + \sqrt{3}i;$$

$$z_3 = 2e^{(\pi/6 + 3\pi/4)i} = 2e^{11\pi i/12} = \frac{\sqrt{2}}{2}(-\sqrt{3} - 1 + i(-1 + \sqrt{3}));$$

$$z_4 = 2e^{(\pi/6 + 4\pi/4)i} = 2e^{7\pi i/6} = -\sqrt{3} - i;$$

$$z_5 = 2e^{(\pi/6 + 5\pi/4)i} = 2e^{17\pi i/12} = \frac{\sqrt{2}}{2}(-\sqrt{3} + 1 - i(1 + \sqrt{3}));$$

$$z_6 = 2e^{(\pi/6 + 6\pi/4)i} = 2e^{5\pi i/3} = -1 + \sqrt{3}i;$$

$$z_7 = 2e^{(\pi/6 + 7\pi/4)i} = 2e^{23\pi i/12} = \frac{\sqrt{2}}{2}(\sqrt{3} + 1 + i(1 - \sqrt{3})).$$