UNIVERSITY OF CALGARY

Faculty of Science
Department of Mathematics & Statistics

MIDTERM #2 - MATH 221 - L11

November 10, 2006

Your family name: ______________________

Your first name: ______________________

Your signature: ______________________

Your student number: ______________________

INSTRUCTIONS:
I. Fill out the above information BEFORE starting this exam.
II. **Show all your work**, use the back of the previous page for rough work and clearly insert the main steps and answers in the provided space.
III. Calculators are not allowed, and no other material.
IV. There are 3 questions and 4 pages to this exam.
V. Time allowed is 50 minutes.

PROBLEM	#1	#2	#3	TOTAL
MARKS	/5	/6	/9	/20

Question 1 (5 points)

[1] a) Explain what it means that a transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is linear.

It means that $T(X+Y) = T(X)+T(Y)$ for all X and Y in $\mathbb{R}^n$, and that $T(aX) = aT(X)$ for all scalars a and and X in $\mathbb{R}^n$.

[1] b) Explain what it means that a transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is a matrix transformation.

It means that there is an $n \times n$ matrix A such that $T(X) = AX$ for all X in $\mathbb{R}^n$.

[2] c) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear, show that T is a matrix transformation and how to find the corresponding matrix A.

If we let $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} a \\ b \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} c \\ d \end{bmatrix}$, then

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = T\left(x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = xT\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + yT\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

So $A = \left[T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)\right]$.

[1] d) Find the matrix of $T : \mathbb{R}^2 \to \mathbb{R}^2$, if T is the reflection in the line $y = -x$.

Since $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, then
the matrix of the projection on $y = -x$ is

$$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Question 2 (6 points)

[1] a) Describe what is a steady-state S of a Markov Chain $S_{m+1} = PS_m$.

A steady-state is simply a probability vector (column adds up to 1) such that $PS = S$

[1] b) What must be one of the eigenvalues of P if there is a steady-state S of the Markov Chain $S_{m+1} = PS_m$? Explain.

Since $PS = S = 1S$ and $S \neq 0$ (since it adds up to 1) then by definition 1 must be an eigenvalue of P.

[3] c) Find all values of a such that $\lambda = 1$ is an eigenvalue of the following matrix

$$P = \begin{bmatrix} .8 & a \\ .2 & 1-a \end{bmatrix}.$$

$\lambda = 1$ is an eigenvalue of P exactly if 1 is a root of the characteristic polynomial $c_A(x) = det(xI - P)$.

But $c_A(1) = det \begin{bmatrix} 1-.8 & -a \\ -.2 & 1-(1-a) \end{bmatrix} = det \begin{bmatrix} .2 & -a \\ -.2 & a \end{bmatrix} = 0$ no matter the value of a.

So $\lambda = 1$ is an eigenvalue of P no matter what a is.

[1] d) Use your result in part c) to find all values of a such that the Markov Chain $S_{m+1} = PS_m$ has a steady-state where P is as in part c).

Since P is a probability matrix, all its entries must be between 0 and 1, so from part c) any $0 \leq a \leq 1$ is a candidate.

We also need the corresponding eigenvectors to be probability vectors, with entries between 0 and 1 and adding up to 1. But in each case we can select the eigenvector $X = \begin{bmatrix} 5a/(1+5a) \\ 1/(1+5a) \end{bmatrix}$.

Question 3 (9 points)
Consider the number of ways x_k to fill a row parking lot with k spaces with Cars taking one space, and Minivans and SUVs each taking 2 spaces.

[2] a) Compute the values of x_k for small $k = 0, 1, 2, 3$.

We have $x_0 = 1$ (one way to do nothing), $x_1 = 1$ (C), $x_2 = 3$ (CC, M, S), $x_3 = 5$ (CCC, CM, CS, MC, SC).

[2] b) Rephrase the question using the technique of dynamical systems.

The values obey the recurrence relation $x_{k+2} = x_{k+1} + 2x_k$. So if we let $V_k = \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}$, then we obtain the dynamical system:

$$V_{k+1} = \begin{bmatrix} x_{k+1} \\ x_{k+2} \end{bmatrix} = \begin{bmatrix} x_{k+1} \\ x_{k+1} + 2x_k \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} = AV_k \text{ and } V_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

[5] c) Describe how you would compute x_k for large values of k with as much detail as possible.

Since $V_k = A^k V_0$, we can try to diagonalize A to compute $V_k = PD^kP^{-1}V_0$.

In fact we find two eigenvalues -1 and 2 with corresponding eigenvectors $X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $X_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

So $P = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$, and therefore $P^{-1} = ı3 \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$, and $P^{-1}V_0 = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$.

We obtain

$$V_k = \frac{1}{3}(-1)^k \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{2}{3}(2)^k \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

from which we get the exact formula:

$$x_k = \frac{2}{3}(2)^k + \frac{1}{3}(-1)^k$$

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