**THE UNIVERSITY OF CALGARY**
DEPARTMENT OF MATHEMATICS AND STATISTICS
**FINAL Handout**
MATH 249

1. Evaluate the limits:

(a) $\lim_{x\to\pi} \dfrac{\cos\left(\dfrac{x}{2}\right)}{\pi - x}$

Since $\cos\left(\dfrac{\pi}{2}\right) = 0$ the type is "$\frac{0}{0}$" so we can use L'Hopital Rule

$$\lim_{x\to\pi} \frac{\cos\left(\dfrac{x}{2}\right)}{\pi - x} = \lim_{x\to\pi} \frac{-\sin\left(\dfrac{x}{2}\right)\cdot\frac{1}{2}}{-1} = \frac{1}{2} \quad (\sin\tfrac{\pi}{2} = 1).$$

(b) $\lim_{x\to-\infty} \dfrac{\cos\left(\dfrac{x}{2}\right)}{\pi - x}$

The type is "$\frac{\text{DNE}}{\infty}$" but $-1 \le \cos\frac{x}{2} \le 1$ and $\pi - x > 0$ so $\frac{-1}{\pi-x} \le \frac{\cos\frac{x}{2}}{\pi-x} \le \frac{1}{\pi-x}$

Since both $\lim_{x\to-\infty} \frac{\pm1}{\pi-x} = 0$ by Squeeze Theorem $\lim_{x\to-\infty} \dfrac{\cos\left(\dfrac{x}{2}\right)}{\pi - x} = 0.$

(c) $\lim_{x\to+\infty} \left(x2^{-x^2}\right)$

Change it to a quotient $\lim_{x\to+\infty} \left(x2^{-x^2}\right) = \lim_{x\to+\infty} \dfrac{x}{2^{x^2}} =$"$\frac{\infty}{\infty}$" $\lim_{x\to+\infty} \dfrac{1}{2^{x^2}\cdot\ln 2\cdot 2x} =$ "$\frac{1}{\infty}$" $= 0$ (by L'Hop.Rule)

(d) $\lim_{x\to-\infty} \dfrac{x}{\sqrt{4x^2+3x+7}}$

The type is "$\frac{-\infty}{\infty}$" but it is not good for L'Hop.Rule

since $\lim_{x\to-\infty} \dfrac{x}{\sqrt{4x^2+3x+7}} = \lim_{x\to-\infty} \dfrac{1}{\frac{1}{2}(4x^2+3x+7)^{-\frac{1}{2}}(8x+3)} =$

$= \lim_{x\to-\infty} \dfrac{2\sqrt{4x^2+3x+7}}{8x+3}$ which is practically the same limit as before.

It is better to divide both top and botttom by the highest power in the denominator $x$

$$\lim_{x\to-\infty} \frac{x}{\sqrt{4x^2+3x+7}} = \lim_{x\to-\infty} \frac{1}{\frac{1}{x}\cdot\sqrt{4x^2+3x+7}} =$$

$= \lim_{x\to-\infty} \dfrac{1}{\frac{-1}{\sqrt{x^2}}\cdot\sqrt{4x^2+3x+7}} =$(since $x = -\sqrt{x^2}$)$= \lim_{x\to-\infty} \dfrac{1}{-\sqrt{4+\frac{3}{x}+\frac{7}{x^2}}} = -\dfrac{1}{2}$

NOTE:

similarily,as $x \to +\infty$ the limit is $\dfrac{1}{2}$ and the graph has two horizontal asymptotes $y = \pm\dfrac{1}{2}$.

2. Find the domain and the derivative of $f$ of

(a) $f(x) = \dfrac{x}{3} e^{-\sin\left(\frac{3}{x}\right)}$
domain is $D = \{x \neq 0\} = (-\infty, 0) \cup (0, +\infty)$
by Product and Chain Rules $\quad f'(x) = \left(\frac{x}{3}\right)' e^{-\sin\left(\frac{3}{x}\right)} + \frac{x}{3} e^{-\sin\left(\frac{3}{x}\right)} \cdot \left(-\sin\frac{3}{x}\right)' =$
$= \frac{1}{3} e^{-\sin\left(\frac{3}{x}\right)} + \frac{x}{3} e^{-\sin\left(\frac{3}{x}\right)} \left(-\cos\frac{3}{x}\right)\left(-\frac{3}{x^2}\right) = e^{-\sin\left(\frac{3}{x}\right)} \left(\frac{1}{3} + \frac{1}{x}\cos\frac{3}{x}\right)$
OR use log.diff $\quad \ln|f| = \ln\left|\frac{x}{3}\right| + \ln e^{-\sin\frac{3}{x}} = \ln|x| - \ln 3 - \sin\frac{3}{x}$
then $\quad \frac{f'}{f} = \frac{1}{x} - 0 - \cos\frac{3}{x} \cdot (3x^{-1})' = \dfrac{1}{x} - \cos\frac{3}{x}(-3)\,x^{-2} = \dfrac{1}{x} + 3x^{-2}\cos\frac{3}{x}$
so $f'(x) = \dfrac{x}{3} e^{-\sin\left(\frac{3}{x}\right)} \left[\dfrac{1}{x} + \dfrac{3}{x^2}\cos\frac{3}{x}\right] = .....$ as above.

(b) $f(x) = \dfrac{\ln(2x-3)}{e^{-x^2}}$
for the domain $\quad 2x - 3 > 0$ so $x > \frac{3}{2}$ and $D = \left]\frac{3}{2}, +\infty\right[$
we can change the function to a product
$f(x) = e^{x^2} \cdot \ln(2x-3)$ then by Product and Chain Rules
$f'(x) = e^{x^2}(2x)\ln(2x-3) + e^{x^2}\frac{1}{2x-3} \cdot 2 = 2e^{x^2}\left(x\ln(2x-3) + \dfrac{1}{2x-3}\right)$

3. **A** $\quad$ Sketch the graph of $y = e^{2x}(6x^2 - 2x - 1)$ i.e.

(a) find the domain, range, vertical and horizontal asymptotes, $x$ and $y$ intercepts;
(b) find the intervals where $f$ is increasing or decreasing;local extrema;
(c) find the intervals where $f$ is concave down or up

**part a)**
domain $D = ]-\infty,, +\infty[ \qquad$ No V.A.
For x-intercepts solve $y = 0 \quad 6x^2 - 2x - 1 = 0 \qquad x = \dfrac{2 \pm \sqrt{28}}{12} = \dfrac{1}{6} \pm \dfrac{\sqrt{7}}{6}$
so $x_1 = -0.27$ and $x_2 = 0.61 \qquad$ for $x = 0$ $y = -1$ y-intercept
For **horizontal asymptotes** $\lim\limits_{x\to+\infty} e^{2x}(6x^2 - 2x - 1) = +\infty$
$\lim\limits_{x\to-\infty} e^{2x}(6x^2 - 2x - 1) = "0 \cdot \infty" = \lim\limits_{x\to-\infty} \dfrac{6x^2 - 2x - 1}{e^{-2x}} = \dfrac{\infty}{\infty}$ L'H. twice
$= \lim\limits_{x\to-\infty} \dfrac{12x - 2}{-2e^{-2x}} = \lim\limits_{x\to-\infty} \dfrac{12}{4e^{-2x}} = \dfrac{1}{\infty} = 0$ thus $y = 0$ is H.A. as $x \to -\infty$.
We will go back to the range later.

**part b)** $\qquad$ by Product Rule
$y' = (e^{2x})'(6x^2 - 2x - 1) + e^{2x}(6x^2 - 2x - 1)' = 2e^{2x}(6x^2 - 2x - 1) + e^{2x}(12x - 2) =$
$= 2e^{2x}(6x^2 - 2x - 1 + 6x - 1) = 4e^{2x}(3x^2 + 2x - 1) = 4e^{2x}(3x - 1)(x + 1)$
for critical points solve $y' = 0 \qquad x = -1, \dfrac{1}{3}$

$f(-1) = \dfrac{7}{e^2}$ $\qquad f(\frac{1}{3}) = e^{\frac{2}{3}} \left[\frac{6}{9} - \frac{2}{3} - 1\right] = -e^{\frac{2}{3}}$

testing $y'$ $\quad - -^{pos} - -_{-1} - -^{neg} -_{\frac{1}{3}} -^{pos} - -$

and the function is **decreasing** on $\left]-1, \dfrac{1}{3}\right[$ **increasing** on $]-\infty, -1[$ and on $\left]\dfrac{1}{3}, \infty\right[$

and at $x = -1$ we got **loc.max** and at $x = \dfrac{1}{3}$ we got **abs.minimum**

thus the **range** $\mathcal{R} = \left[-e^{\frac{2}{3}}, \infty\right[$

**part c)**

$y'' = 4\left(e^{2x}\right)'(3x^2 + 2x - 1) + 4e^{2x}(3x^2 + 2x - 1)' = 4e^{2x}(6x^2 + 4x - 2 + 6x + 2) =$
$= 4e^{2x}(6x^2 + 10x) = 8e^{2x}x(3x + 5)$ solve for possible inflection points $y'' = 0$

$x = 0 \qquad x = -\dfrac{5}{3}$

testing $y''$ $\quad -^{pos} - -_{-\frac{5}{3}} - - -^{neg} - -_{0} - -^{pos} -$

therefore

the function is **concave up** on $\left]-\infty, -\dfrac{5}{3}\right[$ and on $]0, +\infty[$

and it is **concave down** on $\left]-\dfrac{5}{3}, 0\right[$ .

**3B**

Sketch the graph of $y = x(4-x)^3$.Indicate where the function is increasing,decreasing, concave up,concave down;find the domain and range.

**1.step**

function is a polynomial so $D = ]-\infty, +\infty[$ $\qquad$ NO V.A.

"ends": $\lim\limits_{x\to\infty} x(4-x)^3 = +\infty\cdot(-\infty) = -\infty$, $\lim\limits_{x\to-\infty} x(4-x)^3 = -\infty\cdot(+\infty) = -\infty$

NO H.A. $\qquad$ also for $x = 0, 4$ $\qquad y = 0$ intercepts

**2.step**

by Product Rule $y' = 1\cdot(4-x)^3 + x\cdot 3(4-x)^2\cdot(-1) = (4-x)^2(4 - x - 3x) =$
$4(4-x)^2(1-x)$

solve $y' = 0$ for critical points :$x = 1, 4$

testing $y'$ $-^{+} -^{+} -^{+} -_{1} -^{-} -^{-} -^{-} -_{4} -^{-} -^{-} -^{-} -$

the function is incr.on$]-\infty, 1[$ and decr on $]1, +\infty[$ $\qquad f(1) = 27$

and has **abs maximum** at $x = 1$ and horizontal tangents at $x = 1$ and $x = 4$

thus the **range** is $\mathcal{R} = ]-\infty, 27]$

**3.step**

$y'' = 4\cdot 2(4-x)(-1)(1-x) + 4(4-x)^2(-1) = -4(4-x)(2 - 2x + 4 - x) =$
$12(x-4)(2-x)$

for possible inflection points solve $y'' = 0 : x = 4, 2$

testing $y''$ $-^{-} -^{-} -^{-} -_{2} -^{+} -^{+} -_{4} -^{-} -^{-} -^{-} -$

therefore the function is **concave up** on $]2, 4[$, **concave down** on $]-\infty, 2[$ and on $]4, +\infty[$

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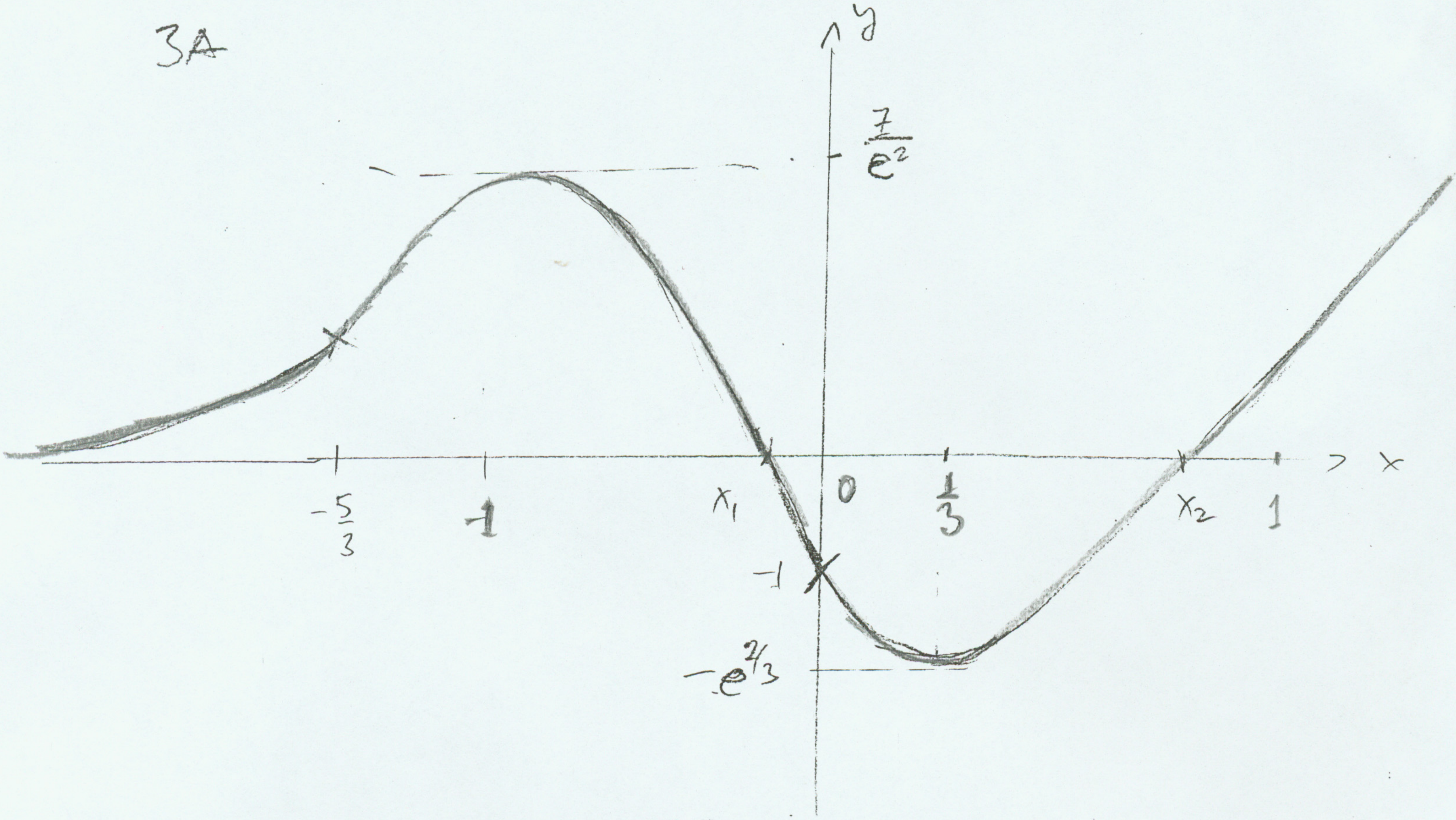
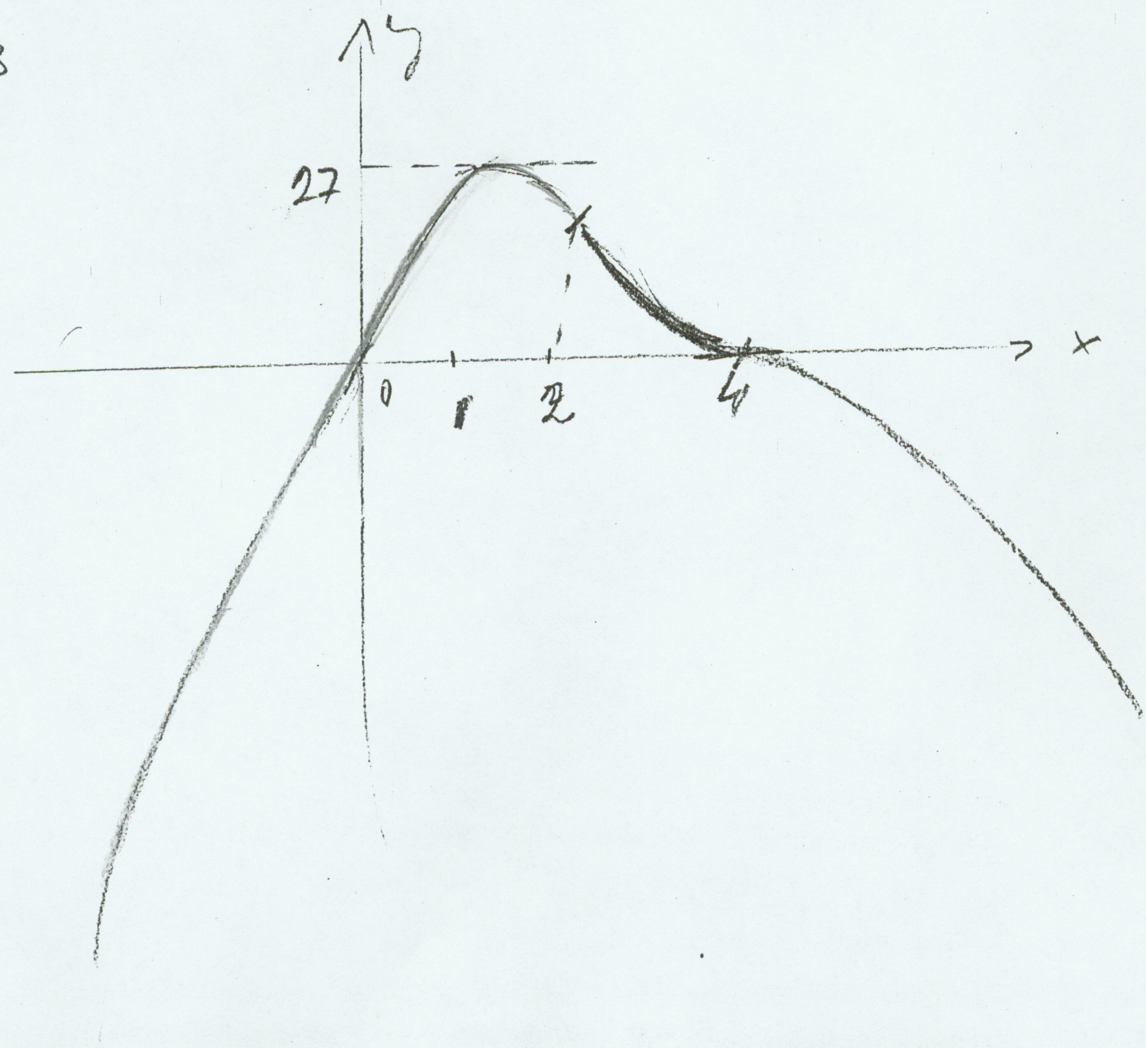
3A




3B



4. (a) Find the tangent approximation (linearization) of

$$f(x) = \frac{1}{\sqrt{2x^2+1}} \text{ around } x_0 = 2.$$

(b) Use it to estimate $\frac{1}{\sqrt{3}}$.

We need $f(2) = \frac{1}{\sqrt{9}} = \frac{1}{3}$ the point is $P\left(2, \frac{1}{3}\right)$

and $f'(x) = \frac{-1}{2}\left(2x^2+1\right)^{-\frac{3}{2}} \cdot 4x = \dfrac{-2x}{\left(\sqrt{2x^2+1}\right)^3}$,

$f'(2) = -\frac{4}{27}$ so the linearization is $L(x) = \frac{-4}{27}(x-2) + \frac{1}{3}$

the approximation equation $\dfrac{1}{\sqrt{2x^2+1}} \doteq \dfrac{-4}{27}(x-2) + \dfrac{1}{3}$ for x close to 2.

To get $\frac{1}{\sqrt{3}}$ we have to $2x^2+1 = 3$ so $x = \pm 1$ but 1 is closer so

substitute $x = 1$ $\frac{1}{\sqrt{3}} \doteq \dfrac{4}{27} + \dfrac{1}{3} = \dfrac{13}{27} = 0.48$

5. **A**

Sketch a graph of <u>one</u> function $f$ satisfying all the following conditions:

(a) $f$ is defined on $]-\infty, +\infty[$, continuous there except

(b) $f$ is discontinuous at $x = 2, 4$ where $\lim_{x\to 4^-} f(x) = f(4) = 0$, $x = 2$ is a vertical asymptote.

(c) $y = 3$ is a horizontal asymptote and $\lim_{x\to -\infty} f(x)$ <u>does not exist</u>,

(d) $f$ is increasing on $]3,4[$ and on $]4,+\infty[$, $f$ is decreasing on $]0,2[$ and on $]2,3[$, and $f'(x) = 0$ for all $x \in ]-2,0[$;

(e) $f$ is concave up on $]0,1[$ and on $]3,4[$; $f$ is concave down on $]1,2[$, on $]2,3[$ and on $]4,+\infty[$;

(f) absolute maximum value is 6, and local minimum value is $-2$.

**B**

(a) Sketch a graph of <u>one</u> function $f$ satisfying all the following conditions:

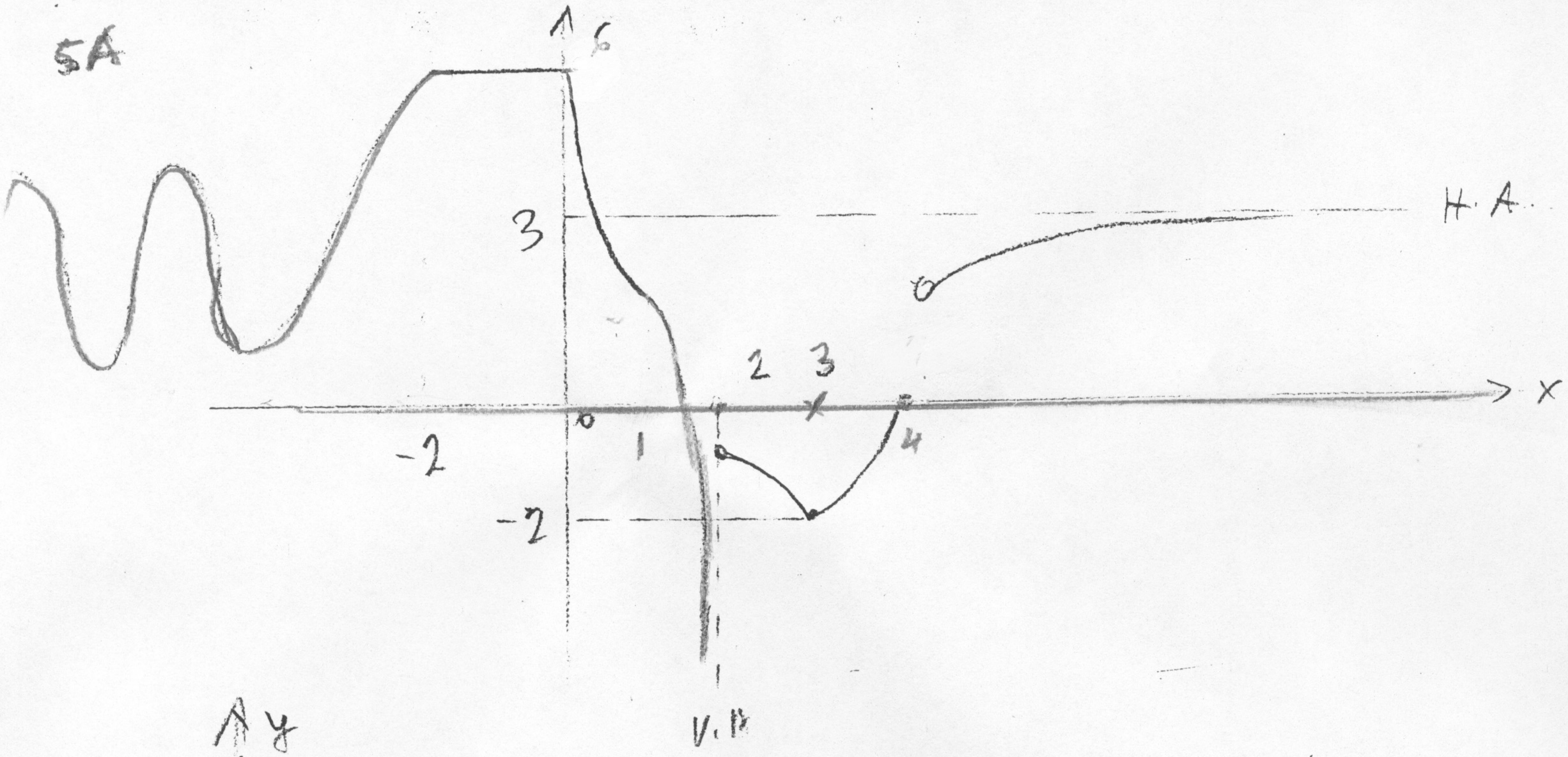
(b) $f$ is defined on $]0,\infty[$

(c) $f$ is discontinuous at $x = 1, 2, 3$ where $\lim_{x\to 2} f(x) = 3$, $\lim_{x\to 3} f(x)$, DNE (does not exist).

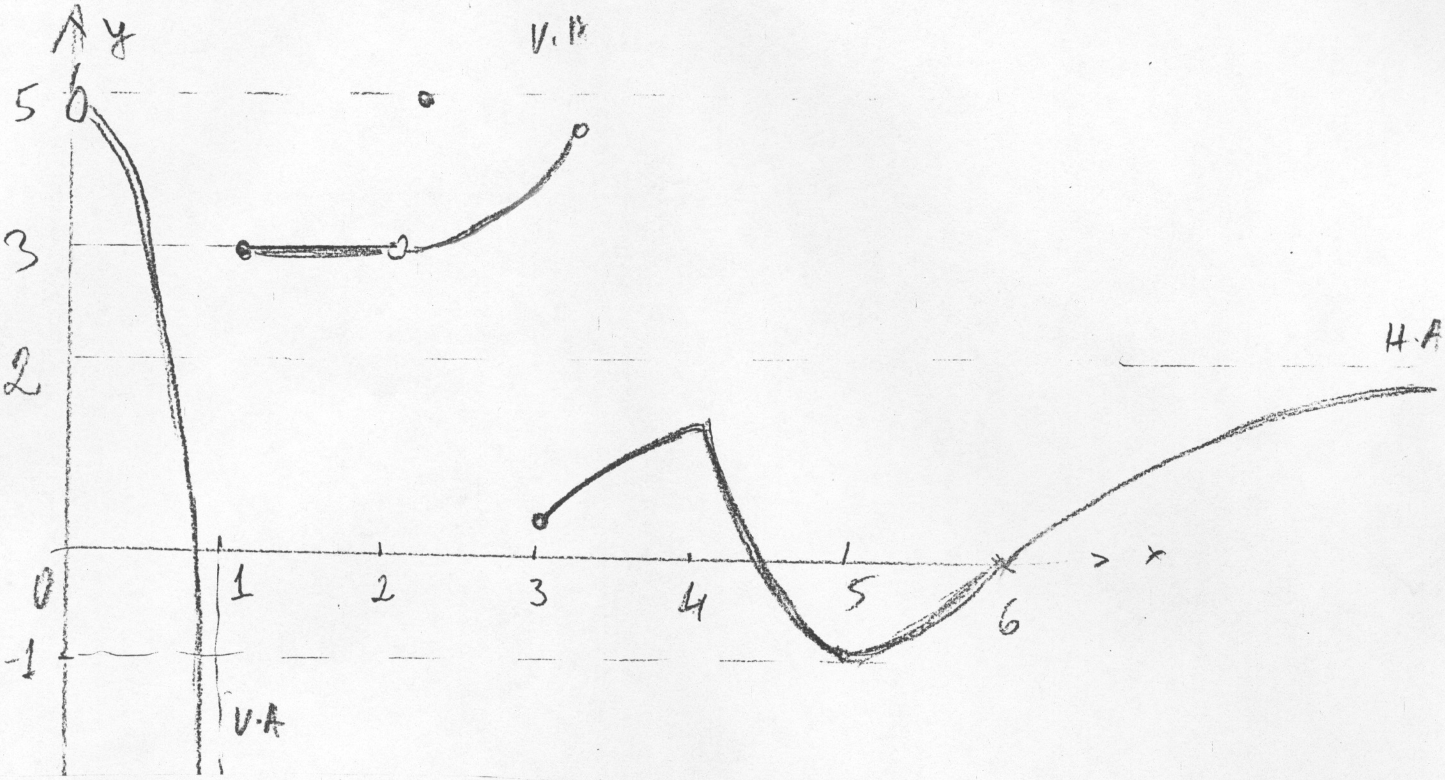
(d) $x = 1$ is V.A., $y = 2$ is H.A.

(e) $f$ is increasing on the intervals $]2,3[\ ]3,4[\ ]5,\infty[$
$f$ is decreasing on $]0,1[$ and on $]4,5[$
$f'(x) = 0$ for all $x \in (]1,2[$

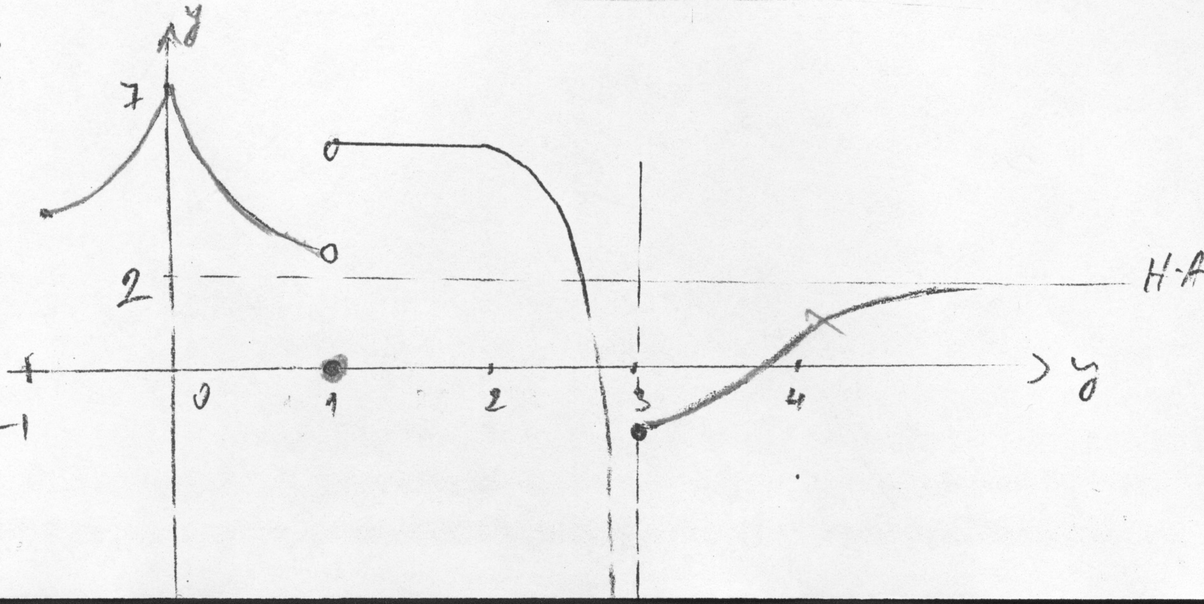
5A




5B




5C



(f) $f$ is concave up on the intervals $]2,3[$ and $]4,6[$, concave down on $]3,4[$ and on $]6,\infty[$

(g) absolute maximum value is 5, local minimum value is $-1$.

**C**

Sketch a graph of <u>one</u> function $f$ satisfying all the following conditions:

(a) $f$ is defined on $[-1,+\infty[$ continuous there except

(b) $f$ is discontinuous at $x=1,3$ where $\lim_{x\to 1} f(x)$ <u>does not exist</u>,.

(c) $x=3$ is a vertical asymptote,and $y=2$ is a horizontal asymptote,

(d) $f$ is increasing on $]-1,0[$ and on $]3,+\infty[$, $f$ is decreasing on $]0,1[$ and on $]2,3[$, and $f'(x)=0$ for all $x\in\,]1,2[$;

(e) $f$ is concave up on $]-1,0[$ ,on $]0,1[$and on $]3,4[$, $f$ is concave down on $]2,3[$ and on $]4,+\infty[$;

(f) absolute maximum value is 7, and local minimum value is 0.

6. **A**

A box with a square base(bottom) and NO top(lid) has a volume of 9 $m^3$.Find the dimensions of the most economical box

if the material for the base costs \$2 per $m^2$ and the material for the sides \$3 per $m^2$.

the length of the side of the base is $x$ and the height is $y$ then the volume $V=x^2y=9$ .....given

looking for min of cost $C = 2\cdot area$ of the base $+3\cdot$ area of sides=

$= 2x^2 + 3\cdot 4xy = 2x^2 + 12xy$

reduce to one variable: $y=\dfrac{9}{x^2}$ so the cost $C(x) = 2x^2 + 12x\cdot\dfrac{9}{x^2} = 2\left(x^2+54x^{-1}\right), x>0$

for critical points $C'(x) = 2\left(2x-54x^{-2}\right) = 4\cdot\dfrac{x^3-27}{x^2} = 0$ means $x^3=27$

and $x=3$ m $\qquad y=1$ m

to justify that we have found minimum $\qquad C''(x) = 4\left(1+54x^{-3}\right) > 0$ for $x>0$

so function is concave up and the critical point is minimum

**B**

A landscape architect plans to enclose a 280 $m^2$ rectangular region in a botanical garden.

She will use shrubs costing \$25.00 per meter along three sides and fencing

costing \$10.00 per meter along the fourth side.

Find the dimensions of the region to minimize the total cost.

Draw a diagram and name the variables: fenced side $x$ ,the other side $y$

the area $A = xy = 280$ is given and we are looking for minimum of the cost

$C = 25\,(x + 2y) + 10x = 35x + 50y$

reduce to one variable : $\quad y = \dfrac{280}{x} \quad$ thus $\quad C(x) = 35x + \dfrac{280 \cdot 50}{x}$ and

$C'(x) = 35 - \dfrac{400 \cdot 35}{x^2}. \qquad$ Solve $C' = 0$

$35\left(1 - \frac{400}{x^2}\right) = 35\left(\dfrac{x^2 - 400}{x^2}\right) = 0 \quad$ so $\quad x^2 = 400 \quad$ and $x = 20\text{m} \quad (x > 0)$

back to $y = \frac{280}{20} = 14$ m

To justify that we have found minimum use $C''(x) = \frac{28000}{x^3} > 0$ so *the* function is concave up

and the critical point is a minimum.

(OR $C' > 0$ for $x > 20$ and $C' < 0$ for $x < 20$)

Thus the dimensions are $20m \times 14m$ with one of the longer sides to be fenced.

7. Evaluate:

   (a) for $x > 0$

   $$\int \frac{3\sqrt{x} - 5}{x\sqrt{x}}dx = 3\int \tfrac{1}{x}dx - 5\int x^{-\frac{3}{2}}dx = 3\ln|x| - 5\cdot(-2)x^{-\frac{1}{2}} + c = 3\ln x + \frac{10}{\sqrt{x}} + c$$

   (b) $\displaystyle\int 2x^3\sqrt{2x^2 + 3}dx$

   by substitution $u = 2x^2 + 3 \quad du = 4xdx \quad \frac{1}{2}du = 2xdx$

   the integral $= \int x^2\sqrt{2x^2 + 3}\cdot 2xdx = \frac{1}{2}\int (?)\,\sqrt{u}du =$

   from the substitution $\qquad \dfrac{u - 3}{2} = x^2$ so

   the integral $= \frac{1}{2}\int \dfrac{u - 3}{2}\sqrt{u}du = \frac{1}{4}\int (u - 3)\,\sqrt{u}du = \frac{1}{4}\int u^{\frac{3}{2}}du - \frac{3}{4}\int u^{\frac{1}{2}}du =$

   $= \frac{1}{4}\cdot\frac{2}{5}u^{\frac{5}{2}} - \frac{3}{4}\cdot\frac{2}{3}u^{\frac{3}{2}} + c =$

   (back to x)$= \frac{1}{10}(2x^2 + 3)^{\frac{5}{2}} - \frac{1}{2}(2x^2 + 3)^{\frac{3}{2}} + c \qquad$ for any $x$.

   (c) $\displaystyle\int \sin\frac{x}{3}\,dx = \frac{-\cos\frac{x}{3}}{\frac{1}{3}} + c = -3\cos\frac{x}{3} + c \qquad$ for any x.

8. (a) by substitution $u = x^2 \quad du = 2xdx \quad \frac{1}{2}du = xdx$ and

| $x$ | $u$ |
|---|---|
| 2 | 4 |
| 3 | 9 |

$$\int_2^3 x2^{x^2}dx = \frac{1}{2}\int_4^9 2^u du = \frac{1}{2}\left[\frac{2^u}{\ln 2}\right]_4^9 = \frac{1}{2\ln 2}\left[2^9 - 2^4\right] = \frac{2^4}{2\ln 2}\left[2^5 - 1\right] = \frac{8}{\ln 2}\cdot 31 = \frac{248}{\ln 2}$$

by substitution $u = 3 - 2x \quad du = -2dx \quad -\frac{1}{2}du = dx$ and

| $x$ | $u$ |
|---|---|
| 0 | 3 |
| 1 | 1 |

(b) $\displaystyle\int_0^1 \frac{4x+3}{3-2x}dx = -\frac{1}{2}\int_3^1 \frac{?}{u}du = \frac{1}{2}\int_1^3 \frac{?}{u}du =$

from the substitution

$2x = 3 - u$ so $4x = 6 - 2u$ and $4x + 3 = 9 - 2u$

therefore the integral

$\displaystyle =\frac{1}{2}\int_1^3 \frac{9-2u}{u}du = \frac{9}{2}\int_1^3 \frac{1}{u}du - \int_1^3 du = \frac{9}{2}\left[\ln|u|\right]_1^3 - [3-1] = \frac{9}{2}\ln 3 - 2.$

(c) $\displaystyle\int_e^{e^2} \frac{1}{x\ln x}dx = \int_1^2 \frac{1}{u}du = \left[\ln|u|\right]_1^2 = \ln 2$

by substitution $u = \ln x \quad du = \dfrac{dx}{x}$ and since $\ln e^k = k$

| $x$ | $u$ |
|---|---|
| $e$ | 1 |
| $e^2$ | 2 |