MATH 249
Midterm Handout

1. Evaluate

$\lim_{x\to\infty}\left(x^2 - x^2\cos\frac{1}{x}\right)$

$=\lim_{x\to\infty} x^2\left(1-\cos\frac{1}{x}\right) = \text{"}\infty\cdot 0\text{"} \qquad u=\frac{1}{x}, u\to 0^+$

so $=\lim_{u\to 0^+}\dfrac{(1-\cos u)}{u^2}\cdot\dfrac{(1+\cos u)}{(1+\cos u)} = \lim_{u\to 0^+}\dfrac{(1-\cos^2 u)}{u^2(1+\cos u)} =$

$$=\lim_{u\to 0^+}\frac{\sin^2 u}{u^2(1+\cos u)} = \lim_{u\to 0^+}\left(\frac{\sin u}{u}\right)^2\cdot\lim_{u\to 0^+}\frac{1}{(1+\cos u)} = \frac{1}{2}.$$

2. Evaluate

(a) $\lim_{x\to 0}\dfrac{\sin x}{x-\pi} = \frac{0}{-\pi} = 0$

(b) $\lim_{x\to\pi}\dfrac{\sin x}{x-\pi} = \text{"}\frac{0}{0}\text{"} = \lim_{u\to 0}\dfrac{\sin(u+\pi)}{u} = \lim_{u\to 0}\dfrac{-\sin u}{u} = -1$
where $x-\pi = u$
and $\sin(u+\pi) = \sin u\cos\pi + \cos u\sin\pi, \cos\pi = -1, \sin\pi = 0$

(c) $\lim_{x\to-\infty}\dfrac{\sin x}{x-\pi} = 0$ by Squeeze Th. since $-1\le\sin x\le 1$ $(x-\pi<0)$

$$\frac{-1}{x-\pi}\ge\frac{\sin x}{x-\pi}\ge\frac{1}{x-\pi} \text{ and } \lim_{x\to-\infty}\frac{\pm 1}{x-\pi} = 0.$$

3. For $y=\dfrac{\cos\pi x}{1-x}$ find an equation of the tangent line at $x=-\dfrac{1}{2}$.

for $x=\dfrac{-1}{2} \qquad y=\dfrac{\cos\left(\pi\frac{-1}{2}\right)}{\frac{3}{2}} = 0$ so the point is $\left(\dfrac{-1}{2},0\right)$

use Q.R. to find $y' = \dfrac{-\pi\sin\pi x\ (1-x) - \cos\pi x\,(-1)}{(1-x)^2}$

at $x=-\frac{1}{2} \qquad \cos\frac{-\pi}{2} = 0, \sin\frac{-\pi}{2} = -1$ so $y=0$ and

$y' = \dfrac{3\frac{\pi}{2}}{\frac{9}{4}} = \dfrac{2\pi}{3} = m$... slope $\qquad$ tangent $y=\frac{2\pi}{3}\left(x+\frac{1}{2}\right)$

4. For $y=\left(\sin\frac{1}{\sqrt{x^4+1}}\right)^3$ find y'.

use Chain Rule twice

$y' = 3\left(\sin\frac{1}{\sqrt{x^4+1}}\right)^2\left(\sin\frac{1}{\sqrt{x^4+1}}\right)' = 3\left(\sin\frac{1}{\sqrt{x^4+1}}\right)^2\left(\cos\frac{1}{\sqrt{x^4+1}}\right)\left((x^4+1)^{-\frac{1}{2}}\right)' =$

$= 3\left(\sin\frac{1}{\sqrt{x^4+1}}\right)^2\left(\cos\frac{1}{\sqrt{x^4+1}}\right)\left(-\frac{1}{2}\right)\left((x^4+1)^{-\frac{3}{2}}\right)4x^3 = \dfrac{-6x^3\sin^2\frac{1}{\sqrt{x^4+1}}\cos\frac{1}{\sqrt{x^4+1}}}{(x^4+1)^{\frac{3}{2}}}.$

5. Show that the function $f(x) = x - 2\sin(\pi x)$ has at least one positive zero i.e. $f(x) = 0$ at least for one $x > 0$.

   SOLUTION

   the function is continuous everywhere and $f(\frac{1}{2}) = \frac{1}{2} - 2\sin\frac{\pi}{2} = -\frac{3}{2} < 0$ and $f(1) = 1 - 2\sin\pi = 1 > 0$

   so by IVT there must be an c between $\frac{1}{2}$ and 1 where $f(c) = 0$.

6. Locate all 3 roots of $p(x) = 2x^3 - 6x^2 + 7$ i.e.

   find 3 intervals each containing one root.Sketch the graph of $y = p(x)$.

   SOLUTION

   the polynomial p is continuous everywhere and $p(0) = 7 > 0$

   and $p(-1) = -2 - 6 + 7 = -1 < 0$ so by IVT(intermediate value theorem)

   there must be one root r_1 between -1 and 0 $\quad r_1 \in]-1, 0[$

   since $p(1) = 2 - 6 + 7 = 3 > 0$ and $p(2) = 16 - 24 + 7 = -1 < 0$

   so by IVT there is another root between 1 and 2 $\quad r_2 \in]1, 2[$

   finally $p(3) = 54 - 54 + 7 = 7 > 0$

   by IVT there is a root $\quad r_3 \in]2, 3[$

7. Find $\quad \sec\theta \quad$ if $\sin\theta = \frac{1}{5}$ and $\frac{\pi}{2} < \theta < \frac{3}{2}\pi$.

   SOLUTION

   $\cos\theta = \pm\sqrt{1 - \sin^2\theta} = \pm\sqrt{1 - \frac{1}{25}} = \pm\sqrt{\frac{24}{25}} = \pm\frac{\sqrt{24}}{5}$ but since θ is in the second quadrant

   cos must be negative and $\sec\theta = \frac{1}{\cos\theta} = -\frac{5}{\sqrt{24}}$.

8. If $\cos\theta = \frac{2}{3}$and $\pi < \theta < 2\pi$ find $\sin\theta$ and then $\sin 2\theta$.

   SOLUTION

   $\sin\theta = \pm\sqrt{1 - \cos^2\theta} = \pm\sqrt{1 - \frac{4}{9}} = \pm\sqrt{\frac{5}{9}} = \pm\frac{\sqrt{5}}{3}$ but since θ is in the forth quadrant sin must be negative

   $\sin\theta = -\dfrac{\sqrt{5}}{3}$. Now $\sin 2\theta = 2\sin\theta\cos\theta = 2 \cdot \left(\frac{-\sqrt{5}}{3}\right) \cdot \frac{2}{3} = -\dfrac{4\sqrt{5}}{9}$.

9. Find the values of a and b so that the function f is continuous everywhere

$$f(x) = \begin{cases} \left(\frac{2}{2x+1} - 3\right)(4x^2 - 1) & \text{for} \quad x < -\frac{1}{2} \\ ax + b & \text{for} \quad -\frac{1}{2} \leq x \leq 2 \\ \cos(-\frac{\pi}{x}) & \text{for} \quad x > 2 \end{cases} .$$

   SOLUTION

   The function is continuous except $x = -\frac{1}{2}$ and $x = 2$

$f(-\frac{1}{2}) = \frac{-1}{2}a + b = \lim_{x\to -\frac{1}{2}^+} f(x)$ and $\lim_{x\to -\frac{1}{2}^-} f(x) = \lim_{x\to -\frac{1}{2}^-} \left(\frac{2}{2x+1} - 3\right)(4x^2-1) = \lim_{x\to -\frac{1}{2}^-} \left(\frac{2-3(2x+1)}{2x+1}\right)(4x^2-1)$

$= \lim_{x\to -\frac{1}{2}^-} \left(\frac{2-6x-3}{2x+1}\right)(2x-1)(2x+1) = \lim_{x\to -\frac{1}{2}^-} (-1-6x)(2x-1) = 2\cdot(-2) = -4$

All 3 numbers must be the same so $\frac{-1}{2}a + b = -4$

Similarly,for $x = 2$

$f(2) = 2a + b = \lim_{x\to 2^-} f(x)$ and $\lim_{x\to 2^+} \cos\frac{-\pi}{x} = \cos\frac{-\pi}{2} = 0$

All 3 numbers must be the same so $2a + b = 0 \to\to b = -2a$

back to $\frac{-1}{2}a + b = -4$ $\quad a\left(\frac{-1}{2} - 2\right) = -4 \Rightarrow a\left(\frac{-5}{2}\right) = -4$

so $a = \frac{8}{5}$ and $b = -2a = \frac{-16}{5}$.

10. Find the values of a and b so that the function f is continuous everywhere

$$f(x) = \begin{cases} \cos(\pi x) - 2\sin\frac{\pi x}{2} & \text{for} \quad x > 3 \\ ax^2 + b & \text{for} \quad 0 \le x \le 3 \\ 6\cdot\dfrac{\sqrt{9-x}-3}{x} & \text{for} \quad x < 0 \end{cases}.$$

SOLUTION

The function is continuous except $x = 3$ and $x = 0$

$f(3) = 9a + b = \lim_{x\to 3^-} f(x)$ and

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \left[\cos(\pi x) - 2\sin\frac{\pi x}{2}\right] = \cos(3\pi) - 2\sin\frac{3\pi}{2} = -1 - 2(-1) = 1$

All 3 numbers must be the same so $9a + b = 1$

Similarly,for $x = 0$

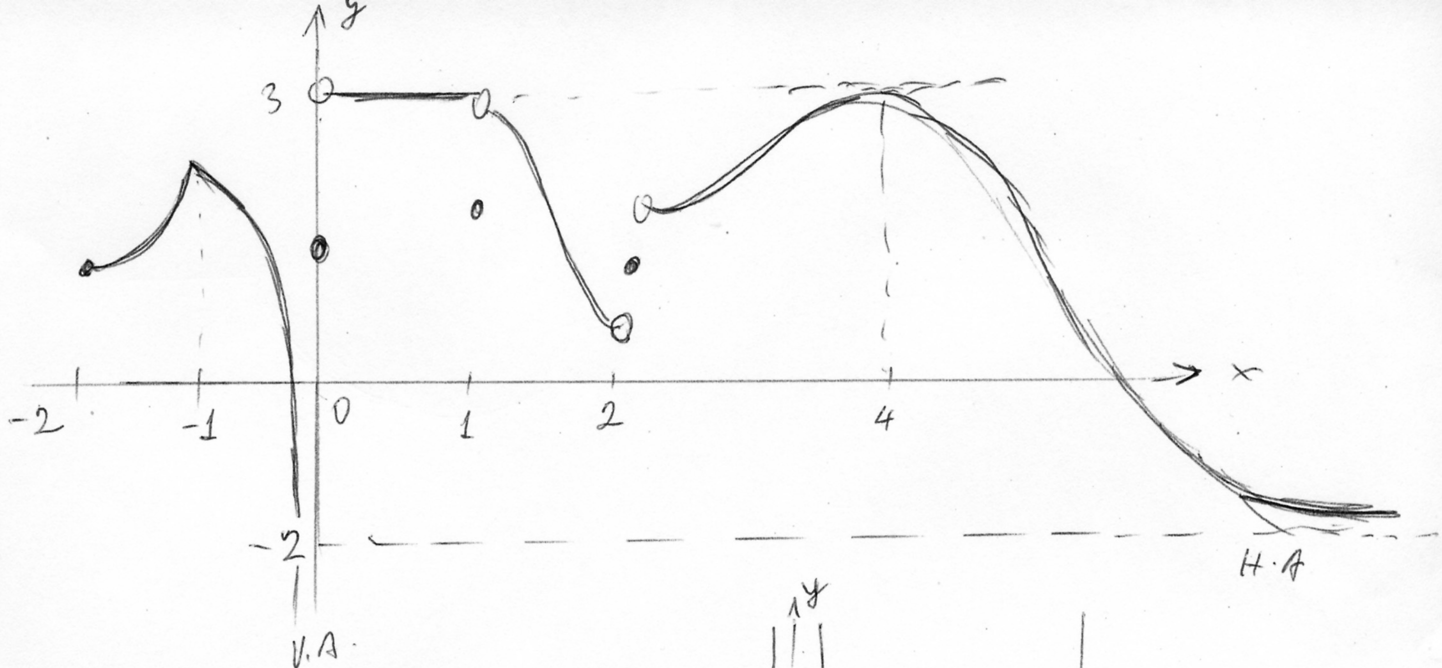
$f(0) = b = \lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0^-} 6\cdot\dfrac{\sqrt{9-x}-3}{x}\cdot\dfrac{\sqrt{9-x}+3}{\sqrt{9-x}+3} = 6\lim_{x\to 0^-}\dfrac{9-x-3^2}{x\left(\sqrt{9-x}+3\right)} =$

$= 6\lim_{x\to 0^-}\dfrac{-x}{x\left(\sqrt{9-x}+3\right)} = 6\lim_{x\to 0^-}\dfrac{-1}{(\sqrt{9-x}+3)} = \frac{-6}{6} = -1$

All 3 numbers must be the same so $b = -1$

substitute $b = -1$ in the first equation $\quad 9a - 1 = 1 \quad a = \frac{2}{9}$ and $b = -1$.

A



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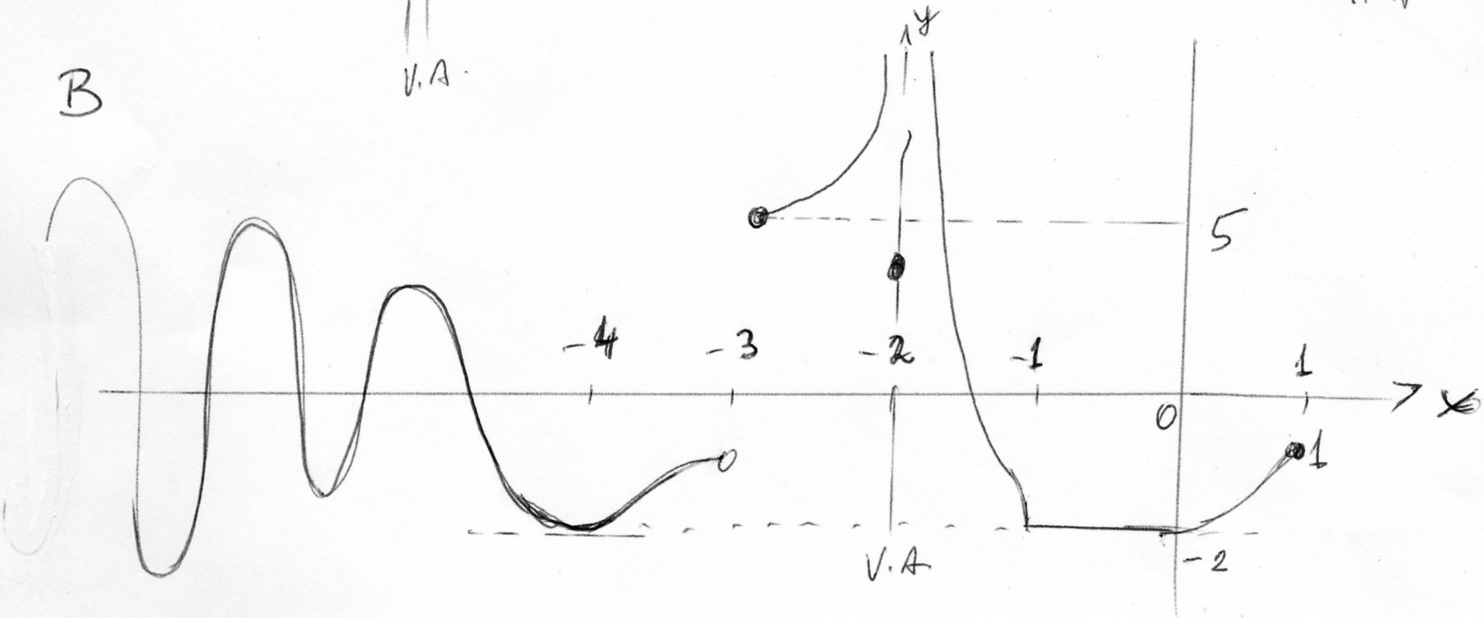

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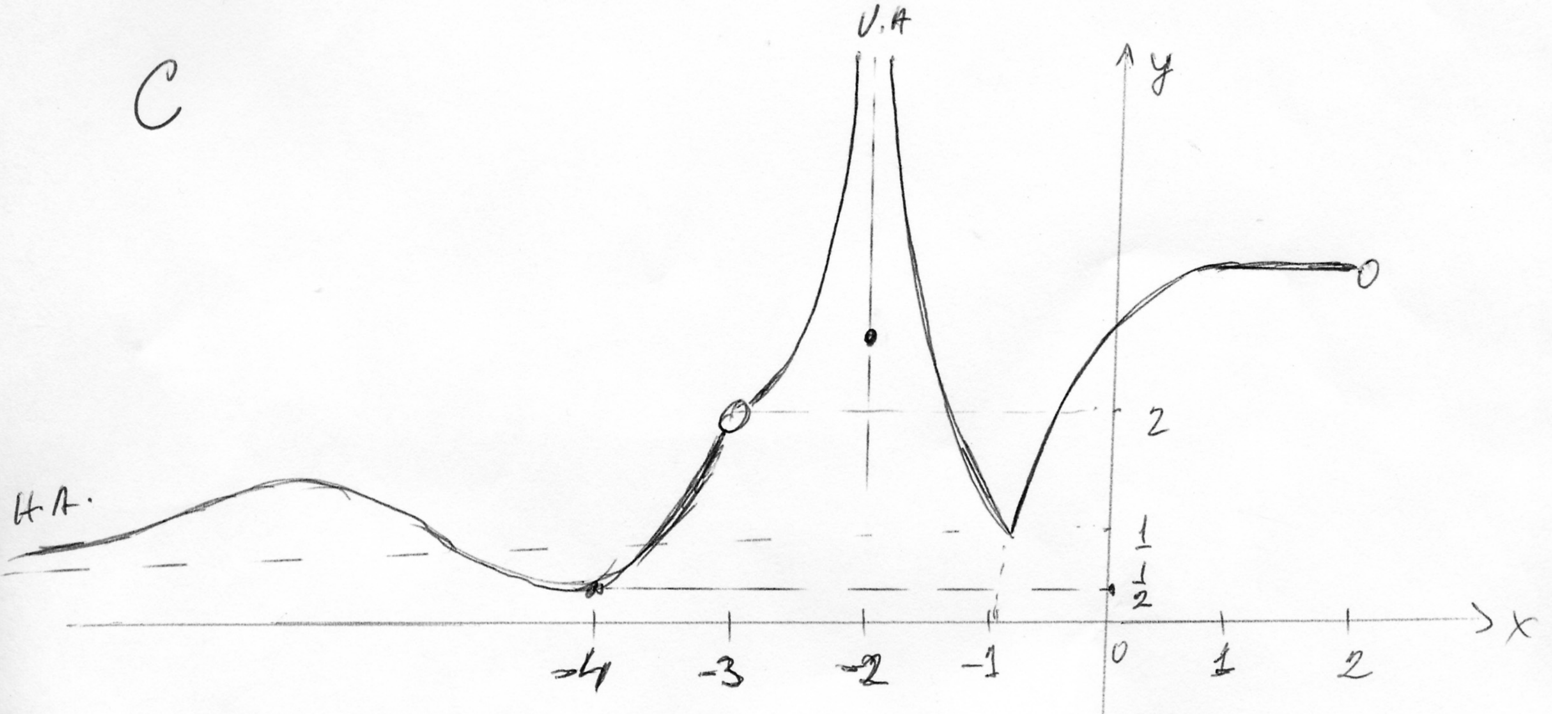

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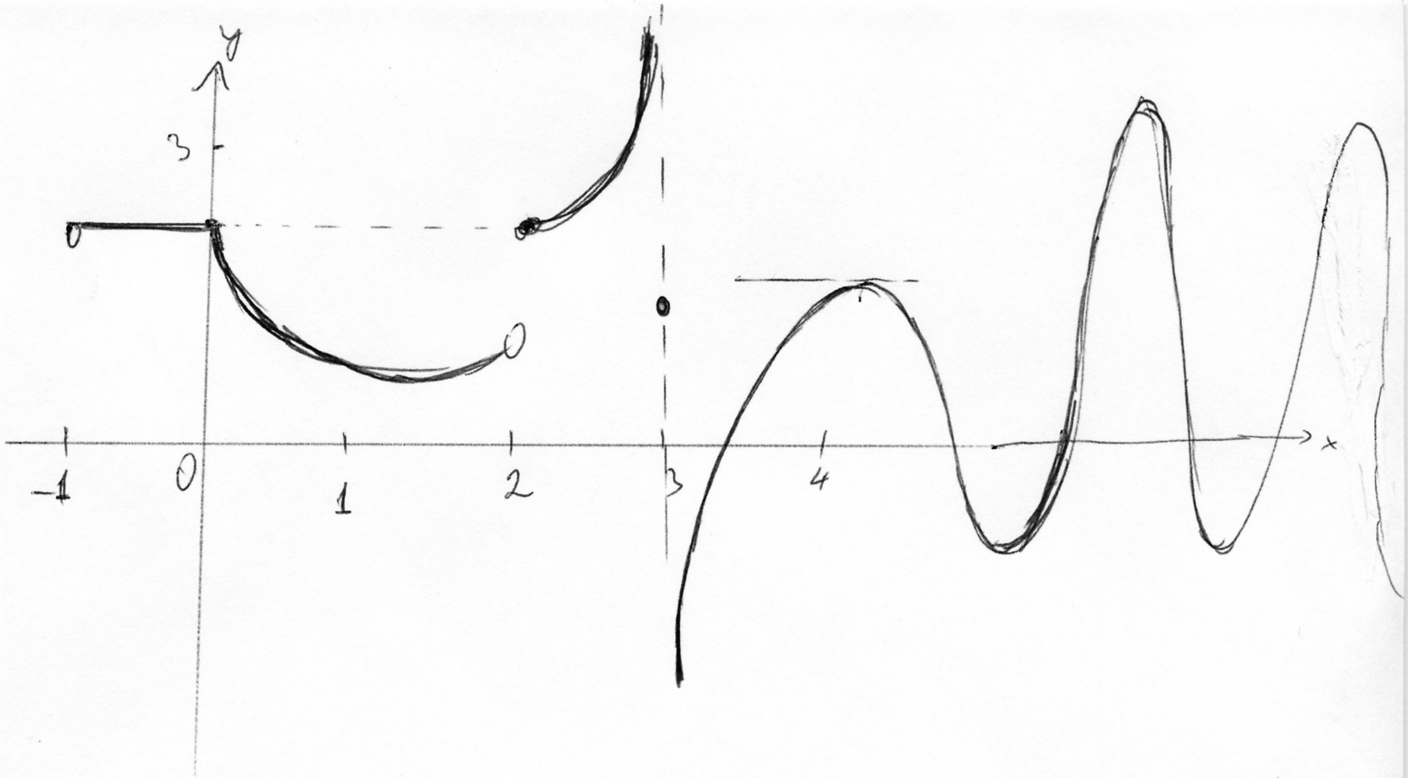

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