THE UNIVERSITY OF CALGARY
DEPARTMENT OF MATHEMATICS AND STATISTICS
FINAL Handout
MATH 251/249

1. Evaluate the limits:

   (a) $\displaystyle\lim_{x\to\pi}\frac{\cos\left(\frac{x}{2}\right)}{\pi-x}$

   Since $\cos\left(\frac{\pi}{2}\right)=0$ the type is "$\frac{0}{0}$" so we can use L'Hopital Rule

   $$\lim_{x\to\pi}\frac{\cos\left(\frac{x}{2}\right)}{\pi-x}=\lim_{x\to\pi}\frac{-\sin\left(\frac{x}{2}\right)\cdot\frac{1}{2}}{-1}=\frac{1}{2}\qquad(\sin\tfrac{\pi}{2}=1).$$

   (b) $\displaystyle\lim_{x\to-\infty}\frac{\cos\left(\frac{x}{2}\right)}{\pi-x}$

   The type is "$\frac{\text{DNE}}{\infty}$" but $-1\le\cos\frac{x}{2}\le 1$ and $\pi-x>0$ so $\frac{-1}{\pi-x}\le\frac{\cos\frac{x}{2}}{\pi-x}\le\frac{1}{\pi-x}$

   Since both $\lim_{x\to-\infty}\frac{\pm1}{\pi-x}=0$ by Squeeze Theorem $\displaystyle\lim_{x\to-\infty}\frac{\cos\left(\frac{x}{2}\right)}{\pi-x}=0.$

   (c) $\displaystyle\lim_{x\to+\infty}\frac{x}{2^x-1}$ the type is "$\frac{+\infty}{+\infty}$" so we can use L'Hop.rule

   $$\lim_{x\to+\infty}\frac{x}{2^x-1}=\lim_{x\to+\infty}\frac{1}{2^x\ln 2}=\text{"}\frac{1}{\infty}\text{"}=0$$

   (d) $\displaystyle\lim_{x\to-\infty}\frac{x}{2^x-1}$ since " $2^{-\infty}$" $=0$

   the limit $\displaystyle\lim_{x\to-\infty}\frac{x}{2^x-1}=\frac{-\infty}{-1}=+\infty.$

   (e) $\displaystyle\lim_{x\to0}\frac{x}{2^x-1}$

   the type is "$\frac{0}{0}$" so we can use L'Hop.Rule again $\displaystyle\lim_{x\to0}\frac{x}{2^x-1}=\lim_{x\to0}\frac{1}{2^x\ln 2}=\frac{1}{\ln 2}$

2. Find the domain and the derivative of f of

   (a) $f(x)=\frac{x}{3}e^{-\sin\left(\frac{3}{x}\right)}$

   domain is $D=\{x\neq0\}=]-\infty,0[\cup]0,+\infty[$

   by Product and Chain Rules $f'(x)=\frac{1}{3}e^{-\sin\left(\frac{3}{x}\right)}+\frac{x}{3}e^{-\sin\left(\frac{3}{x}\right)}\cdot\left(-\cos\frac{3}{x}\right)\cdot\left(-\frac{3}{x^2}\right)=$

   $=e^{-\sin\left(\frac{3}{x}\right)}\left(\frac{1}{3}+\frac{1}{x}\cos\frac{3}{x}\right)$

   OR use log.diff $\ln f=\ln\frac{x}{3}+\ln e^{-\sin\frac{3}{x}}=\ln x-\ln 3-\sin\frac{3}{x}$

   then $\frac{f'}{f}=\frac{1}{x}-0-\cos\frac{3}{x}\cdot(3x^{-1})'=\frac{1}{x}-\cos\frac{3}{x}\cdot(-3)\,x^{-2}=\frac{1}{x}+3x^{-2}\cos\frac{3}{x}$

so $f'(x) = \frac{x}{3} e^{-\sin\left(\frac{3}{x}\right)} \cdot \left[\frac{1}{x} + \frac{3}{x^2}\cos\frac{3}{x}\right] = ...$ as above.

(b) $f(x) = \dfrac{\ln(2x-3)}{e^{-x^2}}$

for the domain $2x - 3 > 0$ so $x > \frac{3}{2}$ and $D = \left]\frac{3}{2}, +\infty\right[$

we can change the function to a product

$f(x) = e^{x^2} \cdot \ln(2x-3)$ then by Product and Chain Rules

$f'(x) = e^{x^2} \cdot 2x \cdot \ln(2x-3) + e^{x^2} \cdot \frac{1}{2x-3} \cdot 2 = 2e^{x^2}\left(x\ln(2x-3) + \dfrac{1}{2x-3}\right)$

3. **A**Sketch the graph of $y = \dfrac{1}{x^2+x-2}$ if $y' = -\dfrac{2x+1}{(x^2+x-2)^2}$and $y'' = \dfrac{6x^2+6x+6}{(x^2+x-2)^3}$ i.e.

(a) find the domain, range, vertical and horizontal asymptotes, x and y intercepts;

(b) find the intervals where f is increasing or decreasing;

(c) find the intervals where f is concave down or up

part a)

domain $D =]-\infty, -2[\cup]-2, 1[\cup (]1, +\infty[$

since the denominator $x^2 + x - 2 = (x+2)(x-1)$ cannot be 0

$x = -2$ and $x = 1$ are V.A.(vertical asymptotes)

since $\lim\limits_{x \to -2^+} \dfrac{1}{(x+2)(x-1)} = " \frac{1}{0^+ \cdot (-3)} " = -\infty$

$\lim\limits_{x \to -2^-} \dfrac{1}{(x+2)(x-1)} = " \frac{1}{0^-(-3)} " = +\infty$ and $\lim\limits_{x \to 1^+} \dfrac{1}{(x+2)(x-1)} = " \frac{1}{3 \cdot 0^+} " = +\infty$

$\lim\limits_{x \to 1^-} \dfrac{1}{(x+2)(x-1)} = " \frac{1}{0^-} " = -\infty$

No x-intercepts since never $y = 0$

For horizontal asymptotes $\lim\limits_{x \to \pm\infty} \dfrac{1}{x^2+x-2} = " \frac{1}{\infty} " = 0$ thus $y = 0$ is H.A. at both ends.

We will go back to the range later.

part b)

given $y' = -\dfrac{2x+1}{(x^2+x-2)^2}$ so obviously $x = -2, 1$ are singular points

and for critical points solve $y' = 0$

so $2x + 1 = 0$ thus $x = -\frac{1}{2}$ is a critical point and $f(-\frac{1}{2}) = \left[\frac{1}{4} - \frac{1}{2} - 2\right]^{-1} = -\frac{4}{9}$

testing $\quad y' \quad -^{+} -^{+} -^{+} -_{-2} -^{+} -^{+} -_{-\frac{1}{2}} -^{-} -^{-} -_{1} -^{-} -^{-} -^{-} -$

$\quad y \quad - -^{incr} - -_{-2} -^{incr} - -_{-\frac{1}{2}} - -^{decr} - -_{1} - -^{decr} --$

and the function is decreasing on the intervals $\left]-\frac{1}{2}, 1\right[$ and $]1, +\infty[$,

it is increasing on $]-\infty, -2[$ and on $\left]-2, -\frac{1}{2}\right[$

part c)

given $y'' = \dfrac{6x^2+6x+6}{(x^2+x-2)^3} = \dfrac{6(x^2+x+1)}{(x+2)^3(x-1)^3}$ for $x \neq -2, 1$
solve for possible inflection points $y''=0$
(never $x^2+x+1=0$ since the discriminant $D = 1-4 = -3$ is negative
always $x^2+x+1>0$

testing y'' $\_^{+}\ \_^{+}\ \_{-2}\ \_^{-}\ \_^{-}\ \_^{-}\ \_^{-}\ \_{1}\ \_^{+}\ \_^{+}\_$
y $\_^{conc.up}\ \_\ \_{-2}\ \_\ \_^{conc.down}\ \_\ \_\ \_{1}\ \_^{conc.up}\ \_^{+}\_$
therefor
the function is concave up on $]-\infty,-2[$ and on $]1,+\infty[$
and it is concave down on $]-2,1[$.
TOGETHER
from the graph we can read the range $R = \left]-\infty,-\frac{4}{9}\right] \cup]0,+\infty[$

3B

Sketch the graph of $y = x(4-x)^3$.Indicate where the function is increasing,decreasing, concave up,concave down;find the domain and range.
1.step
function is a polynoial so $D =]-\infty,+\infty[$
"ends": $\lim\limits_{x\to\infty} x(4-x)^3 = +\infty\cdot(-\infty) = -\infty$, $\lim\limits_{x\to-\infty} x(4-x)^3 = -\infty\cdot(+\infty) = -\infty$
No V. or H. asymptotes
2.step
by Product Rule $y' = 1\cdot(4-x)^3 + x\cdot 3(4-x)^2\cdot(-1) = (4-x)^2(4-x-3x) = 4(4-x)^2(1-x)$
solve $y'=0$ for critical points :$x = 1, 4$

testing y' $\_^{+}\ \_^{+}\ \_^{+}\ \_{1}\ \_^{-}\ \_^{-}\ \_^{-}\ \_{4}\ \_^{-}\ \_^{-}\ \_^{-}\_$
y $\_\ \_^{incr.}\ \_\ \_{1}\ \_\ \_^{decr}\ \_\ \_{4}\ \_\ \_^{incr}\ \_\ \_$
the function is incr.on$]-\infty,1[$ and decr on $]1,+\infty[$ but has a horizonatl tangents at $x=1$ and $x=4$.
also $f(1) = 27$ and $f(4) = 0$
3.step
$y'' = 4\cdot 2(4-x)(-1)(1-x) + 4(4-x)^2(-1) = -4(4-x)(2-2x+4-x) = -4(4-x)(6-3x) =$
$= 12(x-4)(2-x)$
for possible inflection points solve $y''=0$ $x = 4, 2$
testing y'' $\_^{-}\ \_^{-}\ \_^{-}\ \_{2}\ \_^{+}\ \_^{+}\ \_{4}\ \_^{-}\ \_^{-}\ \_^{-}\ \_$
y $\_\ \_^{conc.down}\ \_\ \_{2}\ \_^{conc.up}\ \_\ \_{4}\ \_\ \_^{conc.down}\ \_\ \_$
therefore the function is concave up on $](2,4)[$ and concave down on $]-\infty,2[$ and on $]4,+\infty[$
TOGETHER
we can see that the range is $]-\infty,27[$ and 27 is abs.maxium value at $x=1$.

4. (a) Find the tangent approximation (linearization) of

$$f(x) = \frac{1}{\sqrt{2x^2+1}} \text{ around } x_0 = 2.$$

(b) Use it to estimate $\frac{1}{\sqrt{3}}$.

We need $f(2) = \frac{1}{\sqrt{9}} = \frac{1}{3}$ and $f'(x) = \frac{-1}{2}\left(2x^2+1\right)^{-\frac{3}{2}} \cdot 4x = -2x \cdot \left(\frac{1}{\sqrt{2x^2+1}}\right)^3$,

$f"(2) = -4 \cdot \frac{1}{27}$

so the linearization is $L(x) = \dfrac{1}{3} + \frac{-4}{27}(x-2)$ and

$\frac{1}{\sqrt{2x^2+1}} \doteq \dfrac{1}{3} + \frac{-4}{27}(x-2)$ for x close to 2.

To get $\frac{1}{\sqrt{3}}$ we have to substitute for $x = 1$ so $\frac{1}{\sqrt{3}} \doteq \frac{4}{27} + \frac{1}{3} = \frac{13}{27} = 0.481$.

5. graphs

6. $\mathbb{A}$

A box with a square base(bottom) and NO top(lid) has a volume of 9 m^3.Find the dimensions of the most economical box

if the material for the base costs \$2 per m^2 and the material for the sides \$3 per m^2.

Dimensions: $x \times x \times y$

so the volume $V = x^2y = 9$ .....given

looking for min of cost $C = 2\cdot$area of the base $+3\cdot$ area of sides$= 2x^2 + 3 \cdot 4xy$

reduce to one variable: $y = \dfrac{9}{x^2}$ back to the cost $C(x) = 2x^2 + 12x \cdot \dfrac{9}{x^2} = 2\left(x^2 + 54x^{-1}\right)$

for critical points $C'(x) = 2\left(2x - 54x^{-2}\right) = 4 \cdot \dfrac{x^3-27}{x^2} = 0$ means $x^3 = 27$ and $x = 3m, y = 1m$

to justify that we have found minimum $C''(x) = 2\left(2 + 108x^{-3}\right) > 0$ for $x > 0$

B

A landscape architect plans to enclose a 280 m^2 rectangular region in a botanical garden. She will use shrubs costing \$25.00 per meter along three sides and fencing costing \$10.00 per meter along the fourth side. Find the dimensions of the region to minimize the total cost.

Draw a diagram and name the dimensions $x \times y$

given $A = xy = 280$ and we are looking for minimum of the cost

$C(x) = 25(x+2y) + 10x = 35x + 50y$

reduce to one variable: $y = \dfrac{280}{x}$ thus $f(x) = 35x + \frac{280 \cdot 50}{x}$ and $f'(x) = 5(7 - \dfrac{2800}{x^2})$

Solve $f' = 0$

$5\left(7-\frac{7\cdot 400}{x^2}\right)=5\cdot 7\left(\frac{x^2-400}{x^2}\right)=0$ so $x^2=400$ and $x=20m$ $(x>0)$

back to $y=\frac{280}{20}=14m$

To justify that we have found minimum use $f''(x)=\frac{28000}{x^3}>0$ so f is concave up and the critical point is a minimum.

(OR $f'>0$ for $x>20$ and $f'<0$ for $x<20$)

Thus the dimensions are $20m\times 14m$ with one of the longer sides to be fenced.

7. Evaluate:

(a) for $x>0$

$$\int\frac{3\sqrt{x}-5}{x\sqrt{x}}dx=3\int\frac{1}{x}dx-5\int x^{-\frac{3}{2}}dx=3\ln|x|-5\cdot(-2)x^{-\frac{1}{2}}+c=3\ln x+\frac{10}{\sqrt{x}}+c$$

(b) $\int 2x^3\sqrt{2x^2+3}dx$

by substitution $u=2x^2+3$ $\quad du=4xdx$ $\quad \frac{1}{2}du=2xdx$

the integral $=\int x^2\sqrt{2x^2+3}\cdot 2xdx=\frac{1}{2}\int(?)\sqrt{u}du=.$

from the substitution $\frac{u-3}{2}=x^2$ so

the integral $=\frac{1}{2}\int\frac{u-3}{2}\sqrt{u}du=\frac{1}{4}\int(u-3)\sqrt{u}du=\frac{1}{4}\int u^{\frac{3}{2}}du-\frac{3}{4}\int u^{\frac{1}{2}}du=$

$=\frac{1}{4}\cdot\frac{2}{5}u^{\frac{5}{2}}-\frac{3}{4}\cdot\frac{2}{3}u^{\frac{3}{2}}+c=$

(back to x)$=\frac{1}{10}(2x^2+3)^{\frac{5}{2}}-\frac{1}{2}(2x^2+3)^{\frac{3}{2}}+c$ for any x.

8. (a) $\int_2^3 x2^{x^2}dx$

by substitution $u=x^2$ $\quad du=2xdx$ $\quad \frac{1}{2}du=xdx$ and

x	u
2	4
3	9

the integral$=\frac{1}{2}\int_4^9 2^u du=\frac{1}{2}\left[\frac{2^u}{\ln 2}\right]_4^9=\frac{1}{2\ln 2}\left[2^9-2^4\right]=\frac{2^4}{2\ln 2}\left[2^5-1\right]=\frac{8}{\ln 2}\cdot 31=\frac{248}{\ln 2}$

(b) $\int_0^1\frac{4x+3}{3-2x}dx$

by substitution $u=3-2x$ $\quad du=-2dx$ $\quad -\frac{1}{2}du=dx$ and

x	u
0	3
1	1

the integral $=-\frac{1}{2}\int_3^1\frac{?}{u}du=\frac{1}{2}\int_1^3\frac{?}{u}du=$

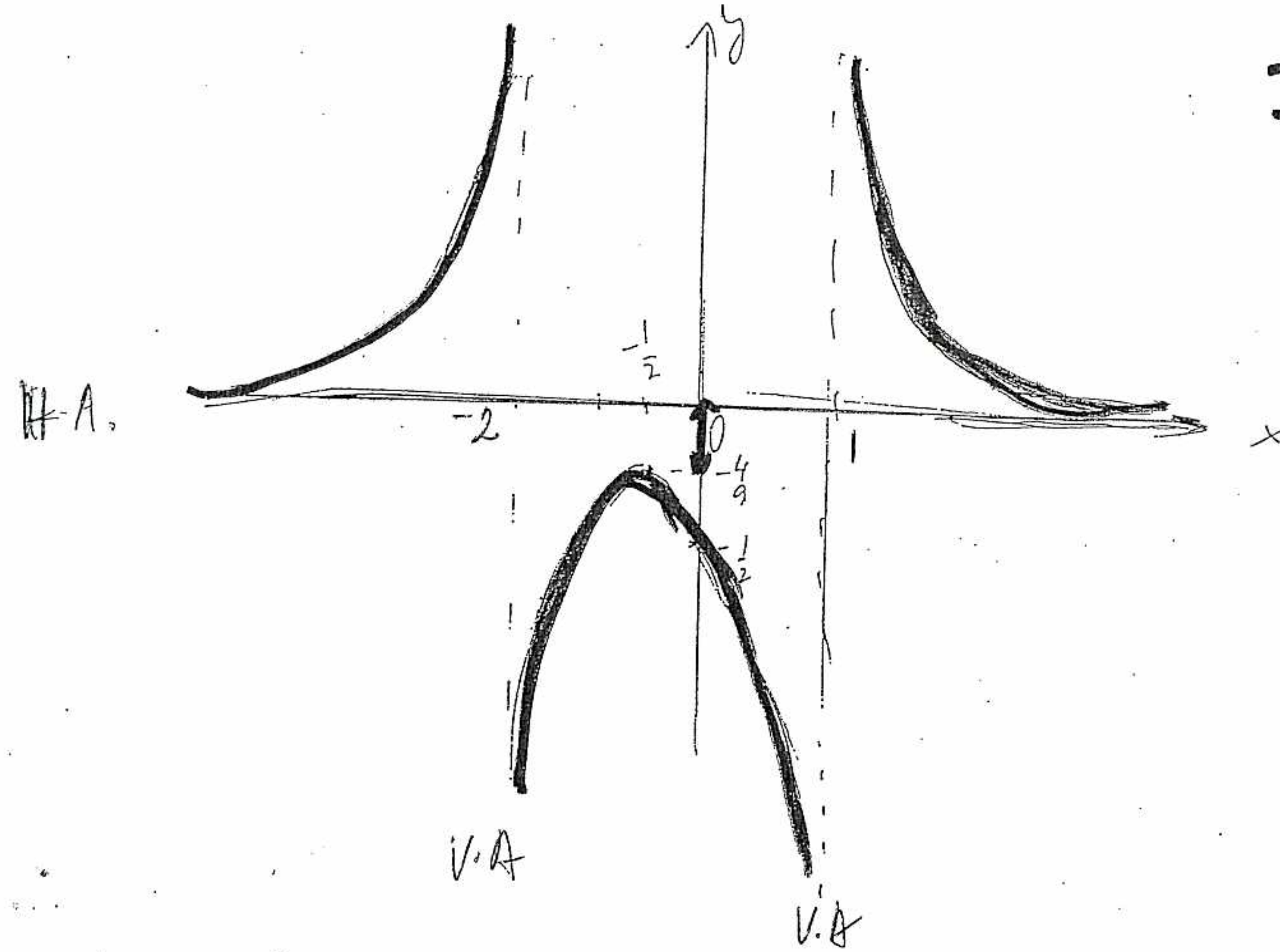
from the substitution

$2x=3-u$ so $4x=6-2u$ and $4x+3=9-2u$

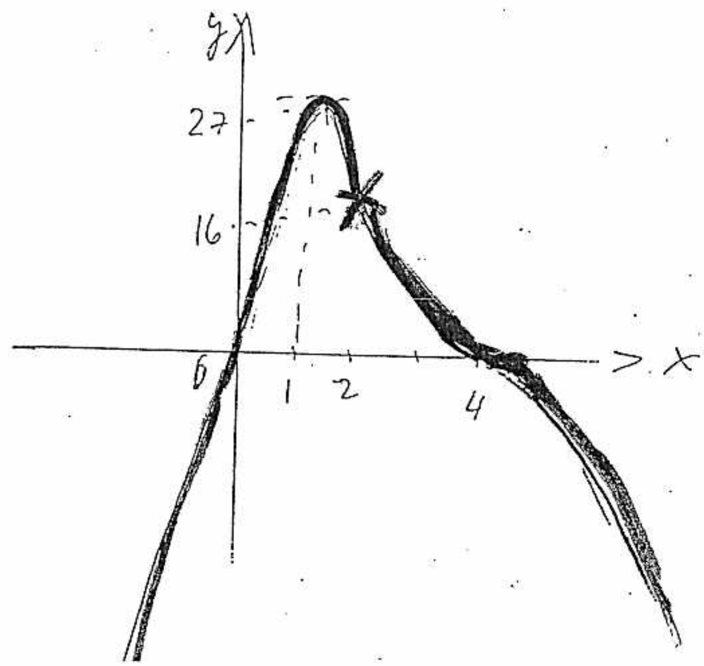
therefore the integral $=\frac{1}{2}\int_1^3\frac{9-2u}{u}du=\frac{9}{2}\int_1^3\frac{1}{u}du-\int_1^3 du=\frac{9}{2}\left[\ln|u|\right]_1^3-[3-1]=$

$=\frac{9}{2}\ln 3-2$

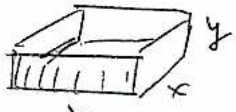
3A



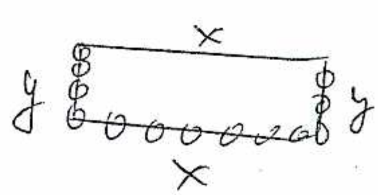
3B



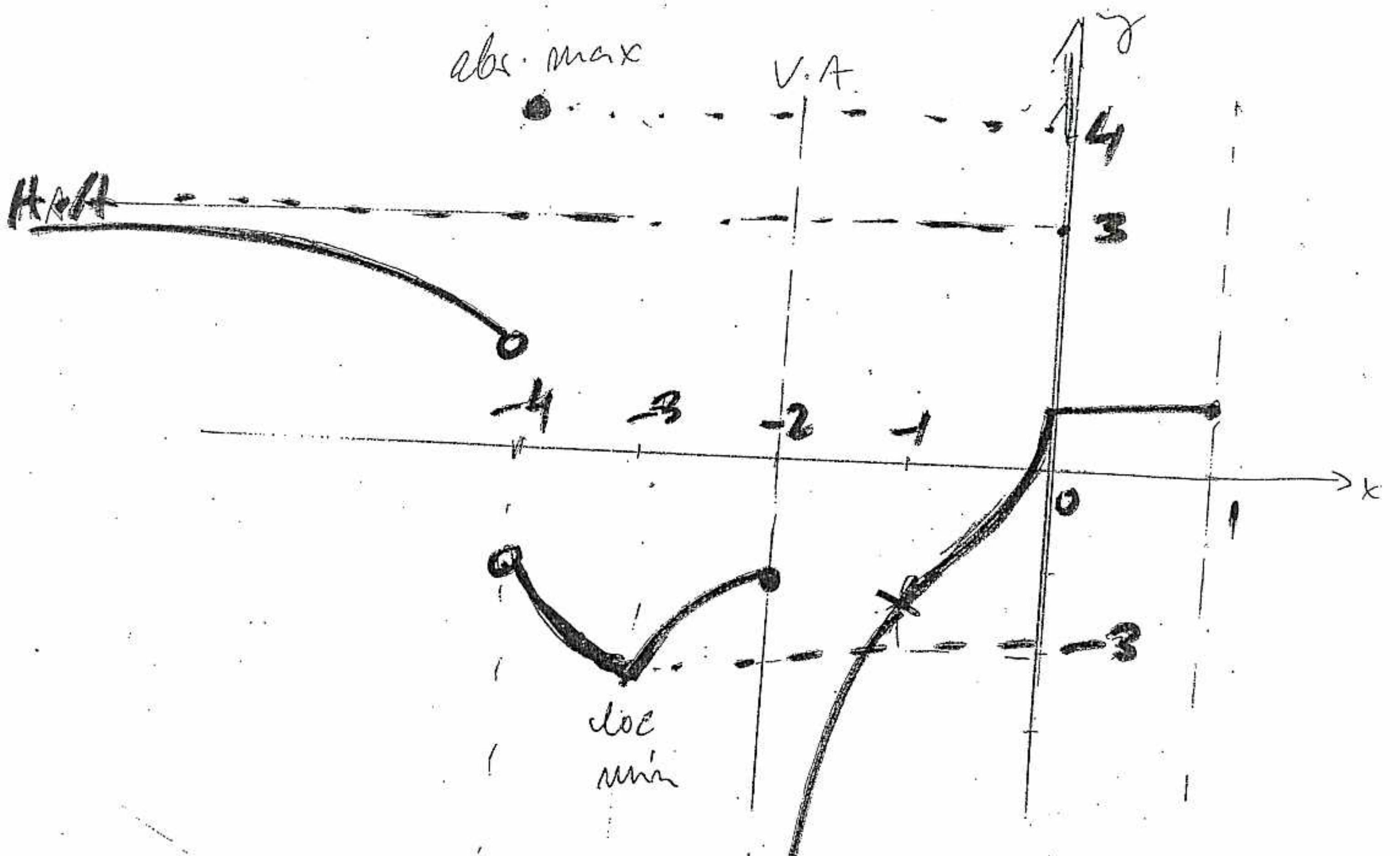
6A



6B



4B


abs. max
V.A
H.A
y
4
3
-4
-3
-2
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0
1
x
-3
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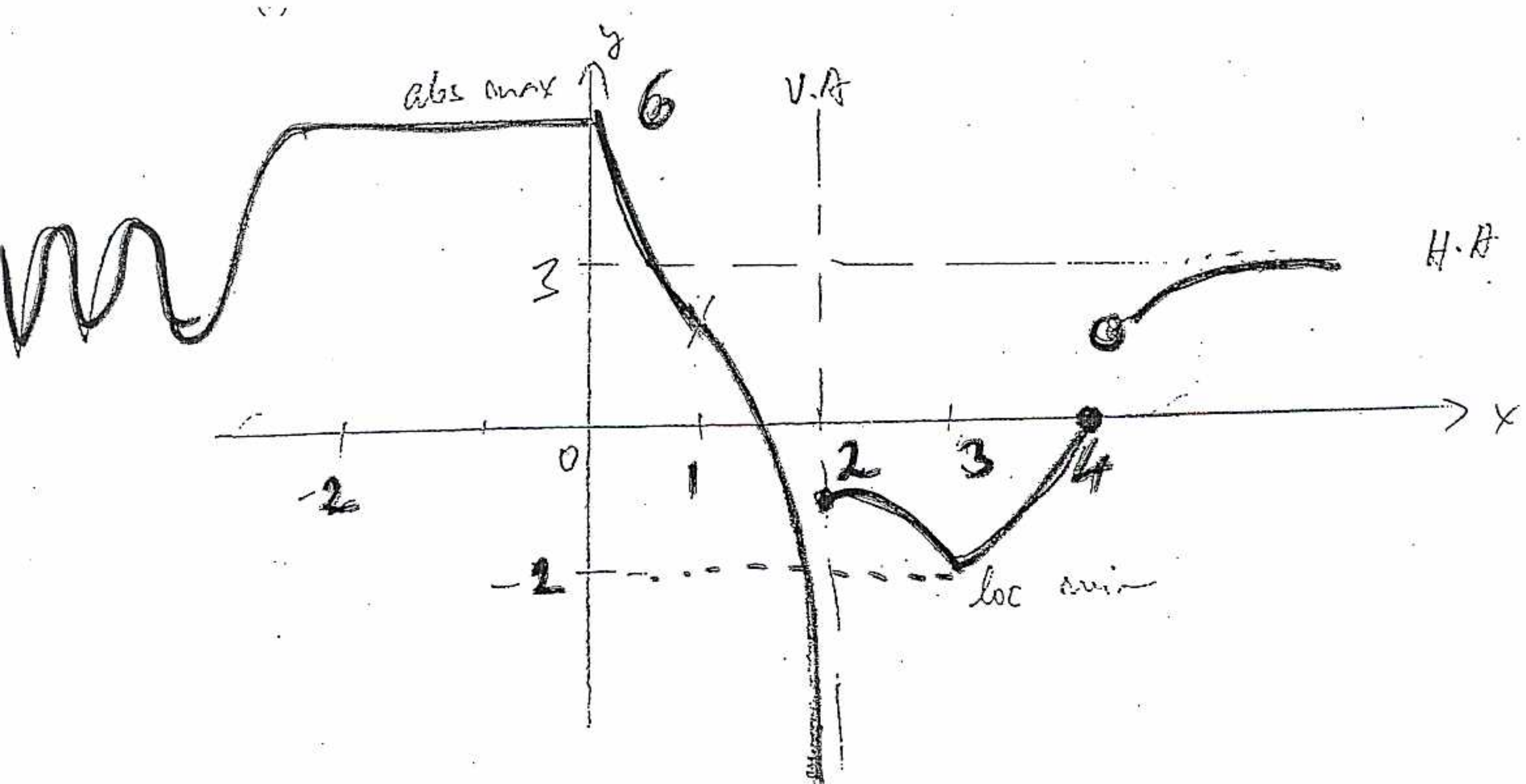
4A


y
abs max
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V.A
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