MATH 249(L 03)
Midterm

SOLUTION

1. Calculate

(a) $\lim_{x\to 1}\dfrac{\sin(3x-3)}{1-x^2} = "\dfrac{0}{0}"(L'H.R) = \lim_{x\to 1}\dfrac{\cos(3x-3)\cdot 3}{-2x} = \dfrac{3}{-2} = -\dfrac{3}{2};$

OR

$\lim_{x\to 1}\dfrac{\sin(3x-3)}{1-x^2} = \lim_{x\to 1}\dfrac{\sin(3x-3)}{3x-3}\cdot\dfrac{(3x-3)}{1-x^2} = \lim_{x\to 1}\dfrac{\sin(3x-3)}{3x-3}\cdot\dfrac{3(x-1)}{(1-x)(1+x)} =$

$= \lim_{h\to 0}\dfrac{\sin h}{h}\cdot\lim_{x\to 1}\dfrac{-3}{1+x} = 1\cdot\left(-\frac{3}{2}\right) = -\dfrac{3}{2}$ where $h = 3x-3$

(b) $\lim_{x\to+\infty}\dfrac{\sin(3x-3)}{1-x^2} = 0$ by Squeeze Theorem since $-1 \le \sin(..) \le 1$ and

($1-x^2$ neg) $-\dfrac{1}{1-x^2} \ge \dfrac{\sin(3x-3)}{1-x^2} \ge \dfrac{1}{1-x^2}$ and $\lim_{x\to+\infty}\dfrac{\pm 1}{1-x^2} = 0.$

2. Find $f'(x)$ of $f(x) = \dfrac{\sin(x^3-2x-\frac{\pi}{4})}{\sqrt{2x+1}}$

by Pr.OR Qu.and Chain Rules

$f'(x) = \left(\sin(x^3-2x-\frac{\pi}{4})\right)'(2x+1)^{-\frac{1}{2}} + \sin(x^3-2x-\frac{\pi}{4})\left[(2x+1)^{-\frac{1}{2}}\right]' =$

$= \cos(x^3-2x-\frac{\pi}{4})\left(x^3-2x-\frac{\pi}{4}\right)'(2x+1)^{-\frac{1}{2}} + \sin(x^3-2x-\frac{\pi}{4})\left[-\frac{1}{2}(2x+1)^{-\frac{3}{2}}\cdot 2\right] =$

$= \cos(x^3-2x-\frac{\pi}{4})(3x^2-2)(2x+1)^{-\frac{1}{2}} - \sin(x^3-2x-\frac{\pi}{4})(2x+1)^{-\frac{3}{2}}$

now, $f(0) = \frac{\sin(-\frac{\pi}{4})}{1} = -\dfrac{1}{\sqrt{2}}$ so the point is $\left(0, -\frac{1}{\sqrt{2}}\right)$and the slope is

$m = f'(0) = \cos\left(\frac{-\pi}{4}\right)\cdot(-2) - \sin\left(\frac{-\pi}{4}\right) = \frac{-2}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \dfrac{-1}{\sqrt{2}}$

an equation $y = -\frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}$ or $\sqrt{2}y + x = -1$

3. For $f(x) = x^3 - 12x^2 + 5$

test some values $f(0) = 5$ positive $f(1) = -6$ negative

f is continuous function everywhere so we can use IVT-Intermediate Value Theorem on $[0,1]$

and there is a root $r_1 \in]0,1[$

also $\lim_{x\to+\infty} f(x) = +\infty$ $\lim_{x\to-\infty} f(x) = -\infty$ so it has to be one root $r_2 < 0$

another root $r_3 > 1$ **3 real roots**

Also $f(-1) = -1 - 12 + 5 = -8$

so we can use IVT on $[-1,0]$ and there is a root $r_2 \in]-1,0[$

Or try

$f(12) = 12^3 - 12^3 + 5 > 0$ and $f(11) = 11^3 - 12 \cdot 11^2 + 5 = 121\,(-1) + 5 < 0$

so by IVT there is a root $\quad r_3 \in [11, 12]\,;$

Also $\quad f'(x) = 3x^2 - 24x = 3x\,(x-8) = 0$ for $x = 0, 8$ critical points

values $\quad f(0) = 5 > 0 \qquad f(8) = 8^3 - 12 \cdot 8^2 + 5 = 8^2(-4) + 5 < 0...$

4. Find $\quad \cot 2\theta \quad$ if $\sin\theta = -\frac{3}{4}$ and $-\frac{\pi}{2} < \theta < 0$ cos is positve

$\cos\theta = \sqrt{1 - (-\frac{3}{4})^2} = \dfrac{\sqrt{7}}{4} \qquad$ then $\sin 2\theta = 2\sin\theta\cos\theta = -\dfrac{3\sqrt{7}}{8}$

and $\cos 2\theta = (\cos\theta)^2 - (\sin\theta)^2 = \frac{7}{16} - \frac{9}{16} = -\frac{1}{8}$,finally

$\cot 2\theta = \dfrac{\cos 2\theta}{\sin 2\theta} = \frac{-1}{8} \cdot \frac{-8}{3\sqrt{7}} = \dfrac{1}{3\sqrt{7}}.$

5. Find the values of a and b so that the function f is continuous everywhere

$$f(x) = \begin{cases} \sin(\dfrac{\pi}{2}x) & \text{for} \quad x < -5 \\ ax^2 + b & \text{for} \quad -5 \le x \le 3 \\ \cos\left(\dfrac{6\pi}{x}\right) & \text{for} \quad x > 3 \end{cases}$$

function is defined and continuous everywhere except

$x = -5$ and $x = 3$

$f(3) = \lim_{x\to 3^-} f(x) = 9a + b$

$\lim_{x\to -3^+} f(x) = \lim_{x\to -3^+} \cos(\dfrac{6\pi}{x}) = \cos(2\pi) = 1$

so it must $\quad 9a + b = 1$

$f(-5) = \lim_{x\to -5^+} f(x) = 25a + b \qquad \lim_{x\to -5^-} f(x) = \lim_{x\to 1^-} \sin\dfrac{\pi}{2}x = \sin(-\dfrac{5\pi}{2}) = -1$

so it must $\quad 25a + b = -1$

solve for $a, b:$ $\quad 9a + b = 1$

$25a+b = -1 \qquad$ subtract $\qquad 16a = -2$ so $a = -\dfrac{1}{8}, b = 1 - 9a = \dfrac{17}{8}$

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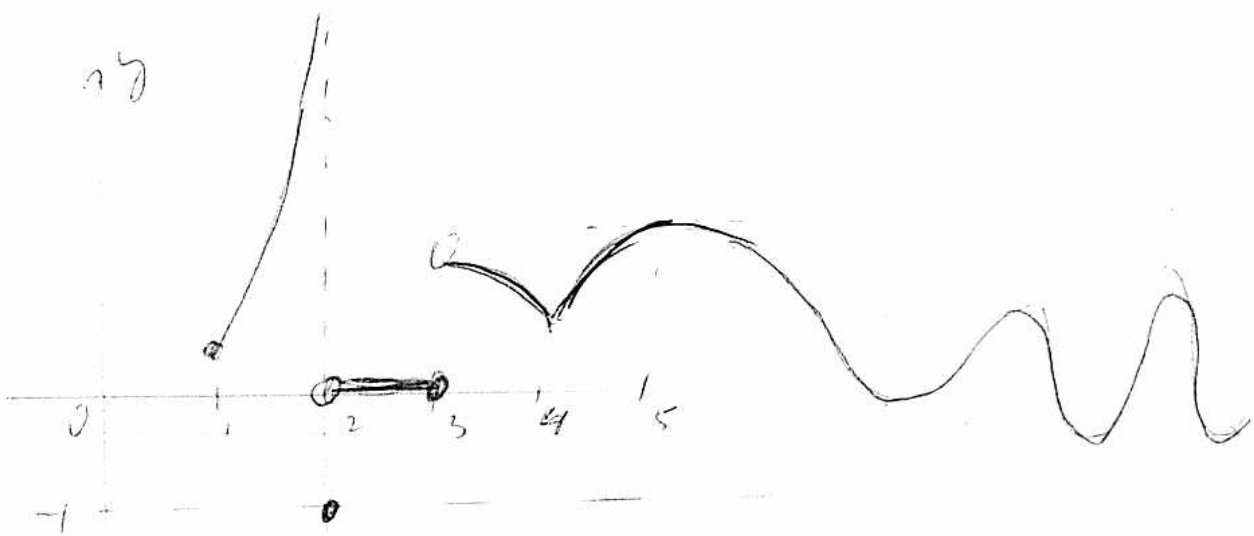

0
1
2
3
4
5
-1