

The University of Calgary
Department of Mathematics and Statistics
MATH 249
Worksheet #5

1. How much money one has to invest today at the interest of 3% compounded quarterly to get \$10,000 in 10 years?
2. Find the domain and derivative of $f(x) = (3x)^\pi + \pi^{3x} + (\pi x)^{3x}$.
3. Find y' if $y = 2^{x^4} + \frac{2}{x^4} + \left(\frac{1}{x}\right)^x$, for $x > 0$.
4. How long does it take to double your investment if the interest of 7 % is compounded
 - (a) yearly?
 - (b) monthly?
5. In the first few weeks after birth, a baby gains weight at a rate proportional to its weight. A baby weighing 4kg at birth weighs 4.4kg after 2 weeks. How much did it weigh 4 days after birth ?
6. For $f(x) = 3^x \ln \frac{3}{x}$ find $f'(3)$.
7. Find y' if $y = x^{x^2} + \ln \frac{1}{1-x}$, for $0 < x < 1$.
8. After 3 days a sample of radon-222 decayed to 58% of its original amount. What is half-life of radon-222?
9. Evaluate

(a)	$\lim_{x \rightarrow +\infty} \frac{x}{2^x - 1}$	(b)	$\lim_{x \rightarrow -\infty} \frac{x}{2^x - 1}$	(c)	$\lim_{x \rightarrow 0} \frac{x}{2^x - 1}$
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10. Evaluate

(a)	$\lim_{x \rightarrow \infty} \frac{x^2}{e^{3x}}$	(b)	$\lim_{x \rightarrow -\infty} \frac{x^2}{e^{3x}}$	(c)	$\lim_{x \rightarrow \infty} \frac{(\ln x)^2}{x}$	(d)	$\lim_{x \rightarrow 0^+} \frac{(\ln x)^2}{x}$.
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SOLUTION

For 1)

the formula to use is $A(t) = A_0 \left(1 + \frac{p}{100n}\right)^{nt}$ where $t = 10, n = 4, p = 3$

$A = 10,000$ $A_0 = ?$ is the initial amount to invest so

$10000 = A_0 \left(1 + \frac{3}{400}\right)^{40} = A_0 \left(\frac{403}{400}\right)^{40}$ multiply by reciprocal to isolate A_0

$$A_0 = 10000 \left(\frac{400}{403}\right)^{40} = \$7416.48$$

For 2) $x > 0$

$$f(x) = (3x)^\pi + \pi^{3x} + (\pi x)^{3x} = 3^\pi x^\pi + (\pi^3)^x + e^{3x \ln \pi x}$$

power + exp.f + chain rule for e^u so

$$\begin{aligned}
f'(x) &= 3^\pi (x^\pi)' + [(\pi^3)^x]' + e^{3x \ln \pi x} [3x \ln(\pi x)]' \text{ (Pr.R.)} \\
&= \pi 3^\pi x^{\pi-1} + (\pi^3)^x \ln \pi^3 + e^{3x \ln \pi x} \left[3 \ln \pi x + 3x \cdot \frac{1}{\pi x} \cdot \pi \right] = \\
&= \pi 3^\pi x^{\pi-1} + 3 (\pi^3)^x \ln \pi + e^{3x \ln \pi x} [3 \ln \pi x + 3]
\end{aligned}$$

ALSO change all terms into e^u and then Chain rule

$$f(x) = (3x)^\pi + \pi^{3x} + (\pi x)^{3x} = e^{\pi \ln 3x} + e^{3x \ln \pi} + e^{3x \ln \pi x}$$

so

$$\begin{aligned}
f'(x) &= e^{\pi \ln 3x} (\pi \ln 3x)' + e^{3x \ln \pi} (3x \ln \pi)' + e^{3x \ln \pi x} (3x \ln \pi x)' = \\
&= e^{\pi \ln 3x} \left(\pi \frac{3}{3x} \right) + e^{3x \ln \pi} (3 \ln \pi) + e^{3x \ln \pi x} \left[3 \ln \pi x + 3x \cdot \frac{1}{\pi x} \cdot \pi \right] = \\
&= \frac{\pi}{x} (3x)^\pi + (3 \ln \pi) \pi^{3x} + [3 \ln \pi x + 3] (\pi x)^{3x}
\end{aligned}$$

For 3)

$$\begin{aligned}
y' &= (e^{x^4 \ln 2})' + (2x^{-4})' + \left(e^{x \ln \frac{1}{x}} \right)' = e^{x^4 \ln 2} (x^4 \ln 2)' - 8x^{-5} + e^{-x \ln x} \cdot (-x \ln x)' = \\
&= 4x^3 \ln 2 \cdot e^{x^4 \ln 2} - 8x^{-5} - e^{-x \ln x} \left(1 \cdot \ln x + x \cdot \frac{1}{x} \right) = 4x^3 \ln 2 e^{x^4 \ln 2} - 8x^{-5} - e^{-x \ln x} (\ln x + 1)
\end{aligned}$$

Or by log.diff.BUT only for the last term $u = \left(\frac{1}{x}\right)^x = e^{x \ln \frac{1}{x}}$ so

$$\ln u = x \ln \frac{1}{x} = x \ln x^{-1} = -x \ln x$$

$$\text{and } \frac{1}{u} \cdot u' = - \left(1 \cdot \ln x + x \cdot \frac{1}{x} \right) = -(\ln x + 1) \text{ so } u' = - \left(\frac{1}{x}\right)^x (\ln x + 1).$$

for the first term we can use $(2^u)' = 2^u \cdot \ln 2 \cdot u'$ so $(2^{x^4})' = 2^{x^4} \cdot \ln 2 \cdot 4x^3$

and the second term is just a power so $(2x^{-4})' = -8x^{-5}$. Together

$$y' = 4x^3 \ln 2 \cdot 2^{x^4} - 8x^{-5} - \left(\frac{1}{x}\right)^x (\ln x + 1) \quad \text{as above.}$$

For 4a)

$$\text{we have to solve for } t \quad 2A_0 = A_0 \left(1 + \frac{7}{100} \right)^t \text{ cancel } A_0$$

$$\text{thus } 2 = (1.07)^t \quad \text{take ln of both sides:}$$

$$\ln 2 = t \ln 1.07, \text{ so } t = \frac{\ln 2}{\ln 1.07} = 10.24$$

We need 10 years and almost 3 months.

$$\text{For 4b) we have to solve for } t \quad 2A_0 = A_0 \left(1 + \frac{7}{1200} \right)^{12t}$$

$$\text{as above thus } 2 = \left(\frac{1207}{1200} \right)^{12t} \text{ and}$$

$$\ln 2 = 12t \ln \frac{1207}{1200}, \text{ and } t = \frac{\ln 2}{12(0.0058163)} = 9.929.$$

So this time we need less than 10 years.

For 5)

for the weight after t weeks $W(t) = 4e^{kt}$, given : $W(2) = 4.4 = 4e^{2k}$

$$\text{solve for } k : \quad \frac{4.4}{4} = e^{2k}, \text{ apply ln to both sides: } \ln 1.1 = 2k, \text{ so } k = \frac{\ln 1.1}{2}$$

Now, after 4 days means $\frac{4}{7}$ of a week, so $W = 4e^{k \frac{4}{7}} = 4e^{\frac{2}{7} \cdot \ln 1.1} = 4.11 \text{ kg.}$

OR $W(t) = 4e^{kt}$ where we measure time t in days then 2 weeks is 14 days

$$W(14) = 4.4 = 4e^{14k} \text{ then } k = \frac{\ln 1.1}{14}$$

$$\text{and for } t = 4 \quad W = 4e^{k4} = 4e^{4 \frac{\ln 1.1}{14}} = (\text{as above}) = 4.11 \text{ kg}$$

For 6)

first simplify $f(x) = 3^x \cdot (\ln 3 - \ln x)$, then use Product Rule

$$f'(x) = (3^x)' (\ln 3 - \ln x) + 3^x (\ln 3 - \ln x)' = 3^x \ln 3 (\ln 3 - \ln x) + 3^x \left(\frac{-1}{x} \right)$$

since $(\ln 3)' = 0$ then $x = 3$

and $f'(3) = 3^3 \ln 3 (\ln 3 - \ln 3) + 3^3 \left(\frac{-1}{3}\right) = -9$.

For 7)

$$y = x^{x^2} + \ln \frac{1}{1-x} = e^{x^2 \ln x} - \ln(1-x) \quad y' = e^{x^2 \ln x} (x^2 \ln x)' - \frac{1}{1-x} (1-x)' =$$

$$= e^{x^2 \ln x} (2x \ln x + x) + \frac{1}{1-x}$$

OR you can use log.diff. but only for the first part

$$u = x^{x^2} \quad \ln u = x^2 \ln x \quad \frac{u'}{u} = 2x \ln x + x^2 \cdot \frac{1}{x} \quad u' = x^{x^2} (2x \ln x + x)$$

and $y' = u' + \frac{1}{1-x}$.

For 8)

the correct formula $A(t) = A_0 e^{kt}$ where $k < 0$, t in days, $A_0 = 100\%$

first info if $t = 3$ $58 = 100e^{3k}$ solve for k $\ln \frac{58}{100} = 3k$

$$k = \frac{\ln 0.58}{3} = -0.1815757$$

so $A(t) = 100e^{kt}$ for k calculated above ;now half-life T means we got 50%

$$50 = 100e^{kT} \quad \text{where } k = \frac{\ln 0.58}{3} \quad \frac{50}{100} = e^{kT}$$

solve for T $\ln 0.5 = kT$ $T = \frac{3 \ln 0.5}{\ln 0.58} = 3.8174$ days

For 9)

for a) $\lim_{x \rightarrow +\infty} \frac{x}{2^x - 1}$ the type is " $\frac{+\infty}{+\infty}$ " so we can use L'Hop.rule

$$\lim_{x \rightarrow +\infty} \frac{(x)'}{(2^x - 1)'} = \lim_{x \rightarrow +\infty} \frac{1}{2^x \ln 2} = \frac{1}{\infty} = 0$$

for b) $\lim_{x \rightarrow -\infty} \frac{x}{2^x - 1}$ since " $2^{-\infty} = 0$ " No L'H.R.

$$\lim_{x \rightarrow -\infty} \frac{x}{2^x - 1} = \frac{-\infty}{-1} = +\infty.$$

for c) $\lim_{x \rightarrow 0} \frac{x}{2^x - 1}$ the type is " $\frac{0}{0}$ " so L'Hop.Rule again

$$\lim_{x \rightarrow 0} \frac{x}{2^x - 1} = \lim_{x \rightarrow 0} \frac{1}{2^x \ln 2} = \frac{1}{\ln 2}$$

For 10)

for a) the type is " $\frac{\infty}{\infty}$ " so use L'Hop.Rule

$$\lim_{x \rightarrow \infty} \frac{x^2}{e^{3x}} = \lim_{x \rightarrow \infty} \frac{2x}{3e^{3x}} \text{ (again)} = \lim_{x \rightarrow \infty} \frac{2}{9e^{3x}} = \frac{2}{\infty} = 0$$

for b) the type is " $\frac{\infty}{0^+}$ " since " $e^{-\infty} = 0$ "

so No L'H.R. but $\lim_{x \rightarrow -\infty} \frac{x^2}{e^{3x}} = \lim_{x \rightarrow -\infty} x^2 e^{-3x} = +\infty$

Or " $\frac{1}{0^+}$ " = $+\infty$.

for c) $\lim_{x \rightarrow \infty} \frac{(\ln x)^2}{x} = \frac{\infty}{\infty}$ (L'H.R.) = $\lim_{x \rightarrow \infty} \frac{2(\ln x) \frac{1}{x}}{1} = \lim_{x \rightarrow \infty} \frac{2 \ln x}{x} =$

again $= \lim_{x \rightarrow \infty} \frac{\frac{2}{x}}{1} = \frac{2}{\infty} = 0$

for d) $\lim_{x \rightarrow 0^+} \frac{(\ln x)^2}{x} = \frac{\infty}{0^+} = \lim_{x \rightarrow 0^+} (\ln x)^2 \cdot \frac{1}{x} = (-\infty)^2 (+\infty) = +\infty$

(No L'H.R.)