THE UNIVERSITY OF CALGARY
DEPARTMENT OF MATHEMATICS AND STATISTICS
FINAL EXAMINATION
Math 249 L(01) - Fall, 2006

Time: 2 hours

NOTE: A calculator *is* allowed.

1. Find $(a)\lim_{x\to 0}\dfrac{1-\cos x}{e^{x^2}-1} = "\frac{0}{0}"$ (L'H.R.)$=\lim_{x\to 0}\dfrac{\sin x}{2xe^{x^2}} = \dfrac{1}{2e^0}\lim_{x\to 0}\dfrac{\sin x}{x} = \dfrac{1}{2}$;

   $(b)\lim_{x\to+\infty}\dfrac{1-\cos x}{e^{x^2}-1} = "\frac{DNE}{\infty}" = 0$

   by Sq.Th.since $-1 \le -\cos x \le 1 \quad 0 \le 1-\cos x \le 2$ so

   $$0 \le \frac{1-\cos x}{e^{x^2}-1} \le \frac{2}{e^{x^2}-1} \text{ and } "\tfrac{2}{\infty}" = 0;$$

   $(c)\lim_{x\to\infty}\dfrac{\ln x}{e^{-x^2}-1} = "\dfrac{\infty}{-1}" = -\infty$ since "$\mathrm{e}^{-\infty}$" $= 0$ (NO L'H.R.)

   for the domain in (a) $\quad 3x+2>0$ and $2-x>0 \to -\frac{2}{3} < x < 2$

   $$D = \left(-\tfrac{2}{3}, 2\right) \text{ and } f'(x) = \left(\frac{\ln(3x+2)}{\sqrt{2-x}}\right)' = \frac{\frac{3}{3x+2}\sqrt{2-x} - \ln(3x+2)\frac{-1}{2\sqrt{2-x}}}{2-x} =$$

   $$= \frac{6(2-x)+(3x+2)\ln(3x+2)}{2\,(3x+2)\,(2-x)\sqrt{2-x}};$$

   (b) for $x>0 \quad g'(x) = (x^{\cos x})' = e^{\cos x \ln x}.\,(\cos x\cdot\ln x)' = e^{\cos x\ln x}.\left(-\sin x\cdot\ln x + \frac{\cos x}{x}\right)$

   or by log.diff. $\quad \ln g(x) = (\cos x\cdot\ln x) \quad \frac{g'}{g} = \left(-\sin x\cdot\ln x + \frac{\cos x}{x}\right)$ then as above.

2. Find a tangent line approximation= linearization of $f(x) = \sqrt[3]{13-5x}$ around $x_0 = 1$;

   (a) then use it to estimate $\sqrt[3]{9}$.

   $$f(1) = \sqrt[3]{8} = 2 \qquad f'(x) = \tfrac{1}{3}\,(13-5x)^{-\frac{2}{3}}\,(-5) = \frac{-5}{3\,(13-5x)^{\frac{2}{3}}}, f'(1) = \frac{-5}{3\cdot\left(\sqrt[3]{8}\right)^2} = -\frac{5}{12}$$

   tangent line: $\quad y = 2 - \frac{5}{12}(x-1) \quad$ and $\sqrt[3]{13-5x} \doteq 2 - \frac{5}{12}(x-1)$ around $x=1$

   we need $\sqrt[3]{9} = \sqrt[3]{13-5x} \quad$ so $9 = 13-5x \quad x = \dfrac{4}{5}$ substitute into approximation equation

   $f(\frac{4}{5}) \doteq L(\frac{4}{5}) \qquad \sqrt[3]{9} \doteq 2 - \frac{5}{12}\left(-\frac{1}{5}\right) = 2 + \frac{1}{12} = 2.08333.$

3. A rectangular box with a square base and a square lid is to hold 12cm^3.

   Find the dimensions of the most economical box if the material for the base costs 4 cents per cm^2,

   and the material for the sides and lid costs 2 cents per cm^2.

   let's call the dimensions of the base $x \times x$ and the height y, then the volum $V = x^2 y = 12$

   thus $\quad y = \dfrac{12}{x^2}$

   the cost $\quad C = 4x^2 + 2(x^2 + 4xy)$ together $f(x) = 4x^2 + 2(x^2 + 4x\frac{12}{x^2}) = 6x^2 + \dfrac{8 \cdot 12}{x}, x > 0$

   now for critical points $f'(x) = 12x - \frac{8 \cdot 12}{x^2} = 12 \cdot \frac{x^3 - 8}{x^2} = 0 \qquad x = 2 \qquad y = \frac{12}{4} = 3$

   $f''(x) = 12 + \frac{8 \cdot 24}{x^3} > 0$ at $x = 2$ so C.P. is a minimum,and dimensions are $2 \times 2 \times 3\ cm.$

4. For $f(x) = \dfrac{x^2}{x-4}$ find

   (a) the domain, vertical and horizontal asymptotes:$\{\ x \neq 4\} = D, x = 4$ is V.A.

   since $\displaystyle\lim_{x \to +4^+} \frac{x^2}{x-4} = +\infty$ and $\displaystyle\lim_{x \to +4^-} \frac{x^2}{x-4} = -\infty$;NO H.A. since $\displaystyle\lim_{x \to ++\infty} \frac{x^2}{x-4} = +\infty$

   and $\displaystyle\lim_{x \to +-\infty} \frac{x^2}{x-4} = -\infty$

   (b) intervals where f is increasing, resp. decreasing:

   $$f'(x) = \left(\frac{x^2}{x-4}\right)' = \frac{2x(x-4) - x^2}{(x-4)^2} = \frac{x^2 - 8x}{(x-4)^2} = \frac{x(x-8)}{(x-4)^2}$$

   $x = 4$ is a sing. point;$x = 0, 8$ are Crit.points

   testing $\qquad f'\ -^{pos} - - -_0 - -^{neg} - - -_4 - -^{neg} - - -_8 -^{pos} - -$

   thus $\qquad f$ is incr. on $(-\infty, 0)$ and on $(8, +\infty)$; it is decr.on $(0, 4)$ and $(4, 8)$

   $f(0) = 0 \qquad f(8) = 16$

   (c) intervals where f is concave up resp. down: $f''(x) = \dfrac{32}{(x-4)^3}$

   $x = 4$ is a sing. point;

   testing $\qquad f''\ -^{neg} - - -_4 - -^{pos} - - -$

   thus f is conc.up on$(4, \infty)$ and conc. down on $(-\infty, 4)$

   (d) all local and absolute extrema, and the range: from b) or c)$x = 0, y = 0$ is loc. max

   and$x = 8$, $y = 16$ is a local mi and we have a gap in the range $R = (-\infty, 0] \cup [16, \infty)$.

5. graph

6. For (a) $\displaystyle\int x\sqrt{3x^2+2}dx = (\text{by subst.}u = 3x^2+2, du = 6dx) = \tfrac{1}{6}\int u^{\frac{1}{2}}du =$

$= \tfrac{1}{6}\cdot\tfrac{2}{3}u^{\frac{3}{2}} + c = \tfrac{1}{9}\left(3x^2+2\right)^{\frac{3}{2}} + c.$

(b) $\displaystyle\int \cos x\sin^2 x\ dx = (\text{ by subst.}u = \sin x, du = \cos x dx) = \int u^2 du = \tfrac{1}{3}\sin^3 x + c.$

7. For(a) by subst., $u = 3x+1, du = 3dx,$ $\begin{array}{cc} x & u \\ 1 & 4 \\ 2 & 7 \end{array}$ and $3x = u-1$

$$\int\limits_1^2 \frac{6x}{3x+1}dx = \tfrac{2}{3}\int\limits_4^7 \frac{u-1}{u}du = \tfrac{2}{3}\int\limits_4^7 \left[1-\frac{1}{u}\right] du = \tfrac{2}{3}\left[u-\ln|u|\right]_4^7 = 2 - \tfrac{2}{3}\ln\tfrac{7}{4};$$

For(b) $\displaystyle\int\limits_1^2 \frac{3x+1}{6x}dx = \int\limits_1^2 \tfrac{1}{2} + \frac{1}{6x}dx = \left[\tfrac{1}{2}x + \tfrac{1}{6}\ln|x|\right]_1^2 = \tfrac{1}{2} + \tfrac{1}{6}\ln 2.$

End of Examination

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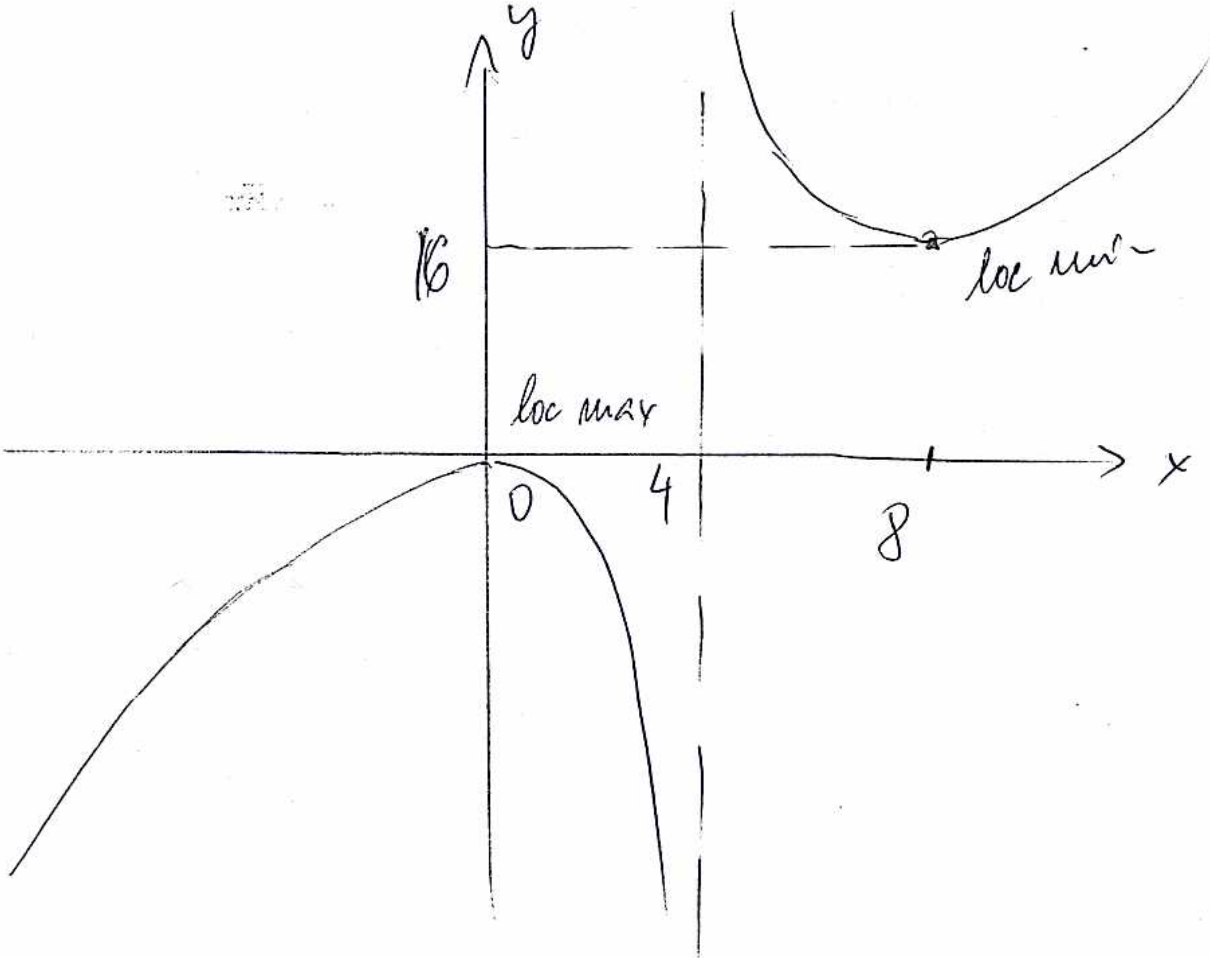

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