

The University of Calgary
 Department of Mathematics and Statistics
 MATH 249
 Worksheet #2

1. For $f(x) = \frac{1}{1-x} \left(1 - \frac{4}{x+3}\right)$ find $\lim f(x)$

- (a) as $x \rightarrow 1$ and
- (b) as $x \rightarrow -3^+$
- (c) as $x \rightarrow +\infty$.

Solution.

For 1a)

the type is " $\frac{0}{0}$ " so we have to simplify

$$f(x) = \frac{1}{1-x} \cdot \frac{x+3-4}{x+3} = \frac{-(1-x)}{(1-x)(x+3)} = \frac{-1}{x+3} \text{ for any } x \neq 1, -3.$$

As $x \rightarrow 1$ the limit is $\frac{-1}{4}$.

For 1b)

We can use the simplification from above and the type of the limit is " $\frac{-1}{0^+}$ " so the limit is $-\infty$.

OR

from the original formula the limit is $L = \frac{1}{4} \cdot \left(1 - \frac{4}{0^+}\right) = -\infty$

For 1c)

From the simplified formula the type is " $\frac{-1}{\infty}$ " so the limit is 0.

OR

from the original the type is " $\frac{1}{-\infty} \cdot \left(1 - \frac{4}{\infty}\right) = 0 \cdot (1 - 0) = 0$.

2. For $f(x) = \sqrt{9-x^2}$ and $g(x) = \frac{3}{x-1}$ find the compositions $g \circ g$ and $f \circ g$ and their domains.

Solution.

For 2)

First the domains of the given functions for D_f solve $9-x^2 \geq 0, (3-x)(3+x) \geq 0$
 split points are $x = \pm 3$, testing $- \text{neg} - -_3 - - \text{pos} - - -_3 - - \text{neg} - - -$

so $D_f = [-3, 3]$ $D_g = \{x \neq 1\}$ since $x-1 \neq 0$.

$$g \circ g(x) = g(g(x)) = \frac{3}{(\dots) - 1} = \frac{3}{\frac{3}{x-1} - 1} = \frac{3}{\frac{3-(x-1)}{x-1}} = \frac{3(x-1)}{4-x}$$

we must start in D_g i.e. $x \neq 1$ and we have to guarantee that

$$4 - x \neq 0 \text{ so } x \neq 4 \quad D_{g \circ g} = \{x \neq 1 \wedge x \neq 4\}$$

$$f \circ g(x) = \sqrt{9 - (\dots)^2} = \sqrt{9 - \frac{9}{(x-1)^2}} = \sqrt{9 \cdot \left(1 - \frac{1}{(x-1)^2}\right)} = 3 \cdot \sqrt{\frac{x^2 - 2x + 1 - 1}{(x-1)^2}} = 3\sqrt{\frac{x(x-2)}{(x-1)^2}}$$

we must start in $D_g, x \neq 1$ and guarantee that $\frac{x(x-2)}{(x-1)^2} \geq 0$, split points are $x = 0, 2, 1$

testing $-$ $-^{pos}$ $-_0$ $-$ $-^{neg}$ $-_{-1}$ $-$ $-^{neg}$ $-_{-2}$ $-$ $-^{pos}$ $-$ $-$ so $D_{f \circ g} = (-\infty, 0] \cup [2, \infty)$.

3. For $g(x) = \frac{4}{2x-8}$ and $f(x) = \sqrt{x^2-9}$ find $g \circ g$ and $g \circ f$ and their domains

Solution

For 3)

$D_g = \{x \neq 4\}, D_f = (-\infty, -3] \cup [3, +\infty)$ since we have to solve: $x^2 - 9 \geq 0, x^2 \geq 9, \sqrt{x^2} = |x| \geq 3$

$$\text{Now, } g \circ g(x) = \frac{4}{2(\) - 8} = \frac{2 \cdot 2}{2 \left[\left(\frac{4}{2x-8} \right) - 4 \right]} = \frac{2}{\frac{4-8x+32}{2x-8}} = 2 \cdot \frac{2x-8}{36-8x} = \frac{4(x-4)}{4(9-2x)} = \frac{x-4}{9-2x}$$

for $x \neq 4$ and $x \neq \frac{9}{2}$, so $D_{g \circ g} = (-\infty, 4) \cup (4, 4.5) \cup (4.5, +\infty)$.

$$\text{For } g \circ f(x) = \frac{4}{2(\) - 8} = \frac{4}{2\sqrt{x^2-9} - 8} = \frac{2}{\sqrt{x^2-9} - 4} \cdot \frac{\sqrt{x^2-9} + 4}{\sqrt{x^2-9} + 4} = \frac{2(\sqrt{x^2-9} + 4)}{x^2 - 9 - 4^2} = \frac{2(\sqrt{x^2-9} + 4)}{x^2 - 25}$$

we know that $x \in D_f$ and that new denominator must be non-zero

$$\sqrt{x^2-9} - 4 \neq 0, \sqrt{x^2-9} \neq 4, \text{ so } x^2 - 9 \neq 4^2, x^2 \neq 25$$

OR after simplification $x^2 - 25 \neq 0$ i.e. $x \neq \pm 5$, together

$$D_{g \circ f} = (-\infty, -5) \cup (-5, -3] \cup [3, 5) \cup (5, +\infty)$$

4. Find : $\lim_{x \rightarrow \infty} \frac{1-4x^2}{6x^2-5x+1}$

(a) as $x \rightarrow -\infty$ (b) as $x \rightarrow \frac{1}{2}$ (c) as $x \rightarrow \frac{1}{3}^-$.

For 4a)

divide top and bottom by x^2 : $\lim_{x \rightarrow -\infty} \frac{\frac{1}{x^2} - 4}{6 - \frac{5}{x} + \frac{1}{x^2}} = \frac{0 - 4}{6 - 0 + 0} = -\frac{4}{6} = -\frac{2}{3}$ (since $\frac{1}{\pm\infty} = 0$)

For 4b)

the type is $\frac{0}{0}$ and we have polynomials

$$\lim_{x \rightarrow \frac{1}{2}} \frac{(1-2x)(1+2x)}{(2x-1)(3x-1)} = \lim_{x \rightarrow \frac{1}{2}} \frac{-(1+2x)}{3x-1} = \frac{-2}{\frac{1}{2}} = -4.$$

For 4c)

we can use the simplification from above but the type is " $\frac{neg\#}{0^-}$ " since $x < \frac{1}{3}$ so $3x-1 < 0$

$$\lim_{x \rightarrow \frac{1}{3}^-} \frac{-(1+2x)}{3x-1} = \frac{-\frac{5}{3}}{0^-} = \frac{1}{0^+} = +\infty$$

5. Find : $\lim_{x \rightarrow 3^+} \frac{\sqrt{3x}-3}{\sqrt{2x^2-6x}}$

(a) as $x \rightarrow 3^+$, (b) as $x \rightarrow +\infty$, (c) as $x \rightarrow 0$

For 5a)

$$\begin{aligned} \lim_{x \rightarrow 3^+} \frac{\sqrt{3x}-3}{\sqrt{2x^2-6x}} &= \frac{0}{0} = \lim_{x \rightarrow 3^+} \frac{\sqrt{3x}-3}{\sqrt{2x^2-6x}} \cdot \frac{\sqrt{3x}+3}{\sqrt{3x}+3} = \lim_{x \rightarrow 3^+} \frac{3x-3^2}{\sqrt{2x^2-6x}} \cdot \frac{1}{\sqrt{3x}+3} = \\ &= \lim_{x \rightarrow 3^+} \frac{3(x-3)}{\sqrt{2x(x-3)}} \cdot \frac{1}{\sqrt{3x}+3} = \lim_{x \rightarrow 3^+} \frac{3}{\sqrt{2x}} \cdot \frac{x-3}{\sqrt{x-3}} \cdot \frac{1}{\sqrt{3x}+3} = \\ &= \lim_{x \rightarrow 3^+} \frac{3}{\sqrt{2x}} \cdot \sqrt{x-3} \cdot \frac{1}{\sqrt{3x}+3} = \frac{3}{\sqrt{6}} \cdot 0 \cdot \frac{1}{6} = 0. \end{aligned}$$

For 5b)

the type is " $\frac{\infty}{\infty}$ " so we have to divide by the highest power in the denominator

in the original from by $x = \sqrt{x^2}$ ($x > 0$) :

$$\lim_{x \rightarrow +\infty} \frac{\sqrt{3x}-3}{\sqrt{2x^2-6x}} \cdot \frac{\frac{1}{\sqrt{x^2}}}{\frac{1}{\sqrt{x^2}}} = \lim_{x \rightarrow +\infty} \frac{\sqrt{\frac{3}{x}-\frac{3}{x}}}{\sqrt{2-\frac{6}{x}}} = \frac{0-0}{\sqrt{2}} = 0$$

OR

we can use the simplified expression from b)

$$\begin{aligned} \lim_{x \rightarrow +\infty} \frac{3}{\sqrt{2x}} \cdot \sqrt{x-3} \cdot \frac{1}{\sqrt{3x}+3} &= \lim_{x \rightarrow +\infty} 3\sqrt{\frac{x-3}{2x}} \cdot \frac{1}{\sqrt{3x}+3} = \\ &= \lim_{x \rightarrow +\infty} 3\sqrt{\frac{1}{2}-\frac{3}{2x}} \cdot \lim_{x \rightarrow +\infty} \frac{1}{\sqrt{3x}+3} = \frac{3}{\sqrt{2}} \cdot 0 = 0 \text{ since } \frac{1}{\infty} = 0. \end{aligned}$$

For 5c)

the limit DNE (does not exist neither as a number nor as $\pm\infty$)

since the function is not defined for small negative $x(\sqrt{neg})$

6. For $f(x) = \frac{\sqrt{3-x}}{x^2-4x+3}$ find $\lim f(x)$

(a) as $x \rightarrow 3^-$ and

(b) as $x \rightarrow 1^+$

(c) as $x \rightarrow +\infty$.

For 6 a)

the type is " $\frac{0}{0}$ " and the function is defined for $x < 3$ and $x \neq 1$ we can simplify

$$f(x) = \frac{\sqrt{3-x}}{x^2 - 4x + 3} = \frac{\sqrt{3-x}}{(x-3)(x-1)} = \frac{\sqrt{3-x}}{-(3-x)(x-1)} = \frac{\sqrt{3-x}}{-\sqrt{3-x}\sqrt{3-x}(x-1)} = \frac{1}{-\sqrt{3-x}(x-1)}$$

Now the type is " $\frac{-1}{0^+ \cdot (2)}$ " = " $\frac{1}{0^-}$ " and the limit is $-\infty$.

For 6b)

We can use the simplification from above or at least identify the type " $\frac{\sqrt{2}}{0}$ " so we have to investigate the sign of the bottom

Since $x > 1$ and $f(x) = \frac{\sqrt{3-x}}{(x-3)(x-1)}$ we can see that the type is " $\frac{\sqrt{2}}{(-2) \cdot 0^+}$ " = " $\frac{1}{0^-}$ " and the limit is $-\infty$.

For 6c)

the limit DNE (does not exist) since the function is not defined for big positive x .

7. For $g(x) = \sqrt{3+x}$ and $f(x) = \sqrt{x-5}$ find the compositions $g \circ g$ and $f \circ g$ and their domains.

For 7)

First the domains of the given functions $D_g = [-3, +\infty)$ since $3+x \geq 0$;

$D_f = [5, +\infty)$ since $x-5 \geq 0$.

$$f \circ g(x) = f(g(x)) = \sqrt{(\cdot) - 5} = \sqrt{\sqrt{3+x} - 5}$$

we must start in D_g i.e. $x \in [-3, +\infty)$ and we have to guarantee that

$$\sqrt{3+x} - 5 \geq 0, \text{ so } \sqrt{3+x} \geq 5$$

both sides are positive so we can square $(3+x) \geq 25$, and $x \geq 22$, together

$$D_{f \circ g} = [22, +\infty[$$

$$g \circ g(x) = \sqrt{3+(\cdot)} = \sqrt{3+\sqrt{3+x}}$$

we must start in $D_g = [-3, +\infty)$ and guarantee that $3 + \sqrt{3+x} \geq 0$

but it is always true for any $x \in [-3, +\infty)$ so $D_{g \circ g} = [-3, +\infty)$.

8. For $f(x) = \frac{|x-3| - |x+3|}{x}$ find $\lim f(x)$

(a) as $x \rightarrow 0$ and

(b) as $x \rightarrow -\infty$

(c) as $x \rightarrow +\infty$.

For 8a)

the type is " $\frac{0}{0}$ " and if x is a small # ,neg. or pos, $x - 3$ is close to -3 so negative and $|x - 3| = -(x - 3) = 3 - x$, $x + 3$ is close to 3 so $x + 3$ is positive and $|x + 3| = x + 3$ therefore

$$f(x) = \frac{|x - 3| - |x + 3|}{x} = \frac{3 - x - (x + 3)}{x} = \frac{-2x}{x} = -2 \text{ for } x \neq 0$$

so the limit is -2 .

ALSO

$$\begin{aligned} f(x) &= \frac{|x - 3| - |x + 3|}{x} \cdot \frac{|x - 3| + |x + 3|}{|x - 3| + |x + 3|} = \frac{|x - 3|^2 - |x + 3|^2}{x \cdot (|x - 3| + |x + 3|)} = \frac{(x - 3)^2 - (x + 3)^2}{x \cdot (|x - 3| + |x + 3|)} = \\ &= \frac{x^2 - 6x + 3^2 - (x^2 + 6x + 3^2)}{x \cdot (|x - 3| + |x + 3|)} = \frac{-12x}{x \cdot (|x - 3| + |x + 3|)} = \frac{-12}{(|x - 3| + |x + 3|)} \text{ for any } x \neq 0 \end{aligned}$$

Now the limit is

$$L = \frac{-12}{3+3} = -2.$$

For8b)

For x big negative number $x - 3$ is negative and $|x - 3| = -(x - 3) = 3 - x$, also $x + 3$ is negative and

$$|x + 3| = -(x + 3) = -3 - x,$$

$$f(x) = \frac{|x - 3| - |x + 3|}{x} = \frac{3 - x + x + 3}{x} = \frac{6}{x} \text{ so the type of the limit is } "\frac{1}{\infty}" \text{ and } L = 0.$$

ALSO for both b) and c)

using the simplification from above $f(x) = \frac{-12}{(|x - 3| + |x + 3|)}$ so the type is " $\frac{-12}{\infty}$ " and the limit is 0 .

For 8 c)

For x big positive both expressions $x - 3$ and $x + 3$ are positive so we can ignore absolute values and

$$f(x) = \frac{x - 3 - (x + 3)}{x} = \frac{-6}{x} \text{ and the type of the limit is } "\frac{-6}{\infty}" \text{ and the limit is } 0.$$

9. For $g(x) = \sqrt{3 - x}$ and $f(x) = \frac{6}{3x - 1}$ find the compositions $g \circ f$ and $f \circ g$

and their domains

For 9)

First the domains of the given functions $D_f = \{x \neq \frac{1}{3}\}$ since $3x - 1 \neq 0$;

$D_g = (-\infty, 3]$ since $3 - x \geq 0$.

$$f \circ f(x) = f(f(x)) = \frac{6}{3(\dots) - 1} = \frac{6}{3 \cdot \frac{6}{3x-1} - 1} = \frac{6}{\frac{18-(3x-1)}{3x-1}} = \frac{6(3x-1)}{19-3x}$$

we must start in D_f i.e. $x \neq \frac{1}{3}$ and we have to guarantee that $19 - 3x \neq 0$ so $x \neq \frac{19}{3}$ and

$$D_{f \circ f} = \left\{ x \neq \frac{1}{3} \wedge x \neq \frac{19}{3} \right\}$$

$$g \circ f(x) = \sqrt{3 - (..)} = \sqrt{3 - \frac{6}{3x-1}} = \sqrt{\frac{3(3x-1)-6}{3x-1}} = \sqrt{\frac{9x-9}{3x-1}} = 3\sqrt{\frac{x-1}{3x-1}}$$

we must start in D_f , $x \neq \frac{1}{3}$ and guarantee that $\frac{x-1}{3x-1} \geq 0$, split points are $x = 1, \frac{1}{3}$

testing $-pos - -\frac{1}{3} - -neg - -1 - -pos -$

so the domain is $D_{g \circ f} = \left(-\infty, \frac{1}{3}\right) \cup [1, +\infty)$