The University of Calgary
Department of Mathematics and Statistics
MATH 249
Worksheet #3

1. Using the **definition of derivative** find $f'(-1)$ if $f(x) = \dfrac{4x}{3-x}$.

2. Find y' if $y = (\dfrac{x^6}{2} - 2x)(4 + \dfrac{1}{\sqrt{2x}})$ for $x > 0$.

3. Find all points on the graph of $y = \dfrac{1}{2x^3 + x^2 + 1}$ where the tangent is horizontal.

4. Using the definition of derivative find $f'(3)$ if $f(x) = \sqrt{\frac{x}{3} + 3}$.

5. Find $f'(-1)$ if $f(x) = (\dfrac{x^3}{6} + \dfrac{1}{2x}).(6 + 2x^2)^{\frac{1}{3}}$.

6. Find all points on the graph of $y = \dfrac{2x}{1+3x}$ where the tangent is parallel to the line $y - 2x = 3$.

7. Using the definition of derivative find $f'(\frac{1}{2})$ if $f(x) = 2x - \dfrac{1}{x}$.

8. Find y' if $y = \sqrt{7x + \dfrac{3}{x^2} + 4\sqrt{x}}$ for $x > 0$.

9. Find an equation of the tangent line to $y = \dfrac{2x-3}{4-2x^5}$ at $x = -1$.

Solution
For 1)

$$f'(-1) = \lim_{h\to 0} \frac{f(-1+h) - f(-1)}{h} = \lim_{h\to 0} \frac{\dfrac{4(-1+h)}{3-(-1+h)} - \dfrac{4(-1)}{3+1}}{h} =$$

$$= \lim_{h\to 0} \frac{\dfrac{4h-4}{4-h} + 1}{h} = \lim_{h\to 0} \frac{1}{h}\left[\frac{4h-4+4-h}{4-h}\right] = \lim_{h\to 0} \frac{1}{h} \cdot \frac{3h}{4-h} = \lim_{h\to 0} \frac{3}{4-h} = \frac{3}{4}$$

(Check by rules $f'(x) = \frac{4}{3-x} + \frac{4x}{(3-x)^2}$ at $x = -1$ $f'(-1) = 1 - \frac{4}{16} = \frac{3}{4}$)

also $$f'(-1) = \lim_{x\to -1} \frac{f(x) - f(-1)}{x - (-1)} = \lim_{x\to -1} \frac{\frac{4x}{3-x} - \frac{-4}{4}}{x+1} = \lim_{x\to -1} \frac{\frac{4x}{3-x} + 1}{x+1} =$$

$$= \lim_{x\to -1} \frac{\frac{4x+3-x}{3-x}}{x+1} = \lim_{x\to -1} \frac{\frac{3(x+1)}{3-x}}{\frac{x+1}{1}} = \lim_{x\to -1} \frac{3(x+1)}{(3-x)(x+1)} = \lim_{x\to -1} \frac{3}{3-x} = \frac{3}{4}$$

For 2)

use Product Rule $$y' = (\frac{x^6}{2} - 2x)'(4 + \frac{1}{\sqrt{2x}}) + (\frac{x^6}{2} - 2x)(4 + \frac{x^{-\frac{1}{2}}}{\sqrt{2}})' =$$

now Power Rule $\quad y' = (\frac{1}{2}6x^5 - 2)(4 + \frac{1}{\sqrt{2x}}) + (\frac{x^6}{2} - 2x)(0 + \frac{1}{\sqrt{2}}\left(\frac{-1}{2}\right)x^{-\frac{3}{2}})$

so $\quad y' = (3x^5 - 2)(4 + \frac{1}{\sqrt{2x}}) + (\frac{x^6}{2} - 2x)(\frac{-1}{2\sqrt{2}}x^{-\frac{3}{2}})$ for $x > 0$

For 3)

slope of a tangent is given by $y' = \left(\frac{1}{2x^3 + x^2 + 1}\right)'$

we can use reciprocal (quotient) or Chain Rule

$y' = \left([2x^3 + x^2 + 1]^{-1}\right)' = (-1)[2x^3 + x^2 + 1]^{-2} . \cdot [2x^3 + x^2 + 1]' =$

$= \frac{(-1)[6x^2 + 2x + 0]}{[2x^3 + x^2 + 1]^2}$

horizontal means slope $m = 0 \qquad$ solve for $x \qquad y' = 0$

a fraction is 0 only if top is 0 $\qquad 6x^2 + 2x = 2x(3x + 1) = 0$

$x = 0 \qquad$ or $\qquad x = -\frac{1}{3}$

at $x = 0, y = 1$ and at $x = -\frac{1}{3}$

$y = \frac{1}{2x^3 + x^2 + 1}\Big|_{x=-\frac{1}{3}} = \left(\frac{-2}{27} + \frac{1}{9} + 1\right)^{-1} = \left(\frac{28}{27}\right)^{-1} = \frac{27}{28}$

at $(0, 1)$ and at $\left(\frac{-1}{3}, \frac{27}{28}\right)$ tangent lines are horizontal.

For 4)

$f'(3) = \lim_{x\to 3}\frac{\sqrt{\frac{x}{3} + 3} - \sqrt{4}}{x - 3} \cdot \frac{\sqrt{\frac{x}{3} + 3} + 2}{\sqrt{\frac{x}{3} + 3} + 2} = \lim_{x\to 3}\frac{\left(\frac{x}{3} + 3\right) - 4}{x - 3} \cdot \frac{1}{\sqrt{\frac{x}{3} + 3} + 2} =$

$= \lim_{x\to 3}\frac{\frac{x}{3} - 1}{x - 3} \cdot \frac{1}{\sqrt{\frac{x}{3} + 3.} + 2} = \lim_{x\to 3}\frac{\frac{1}{3}(x - 3)}{(x - 3)(\sqrt{\frac{x}{3} + 3} + 2)} = \frac{1}{3} \cdot \frac{1}{\sqrt{4} + 2} = \frac{1}{12}$

check by Chain Rule $\qquad f'(x) = \left[\left(\frac{1}{3}x + 3\right)^{\frac{1}{2}}\right]' = \frac{1}{2}\left(\frac{1}{3}x + 3\right)^{-\frac{1}{2}} \cdot \frac{1}{3}$

and $f'(3) = \frac{1}{6\cdot\sqrt{4}} = \frac{1}{12}$

OR

$f'(3) = \lim_{h\to 0}\frac{\sqrt{\frac{(3+h)}{3} + 3} - \sqrt{4}}{h} \cdot \frac{\sqrt{\frac{3+h}{3} + 3} + 2}{\sqrt{\frac{3+h}{3} + 3} + 2} = \lim_{h\to 0}\frac{\frac{(3+h)}{3} + 3 - 4}{h} \cdot \frac{1}{\sqrt{\frac{3+h}{3} + 3} + 2} =$

$= \lim_{h\to 0}\frac{1 + \frac{h}{3} - 1}{h(\sqrt{\frac{3+h}{3} + 3} + 2)} = \frac{\frac{1}{3}}{\sqrt{4} + 2} = \frac{1}{12}$

For 5)

use Product and then Chain Rules for any $x \neq 0$

$f'(x) = (\frac{x^3}{6} + \frac{1}{2x})'.(6 + 2x^2)^{\frac{1}{3}} + (\frac{x^3}{6} + \frac{1}{2x}).\left[(6 + 2x^2)^{\frac{1}{3}}\right]' =$

$= \left[\frac{1}{6} \cdot 3x^2 + \frac{1}{2}\left(-x^{-2}\right)\right].(6 + 2x^2)^{\frac{1}{3}} + (\frac{x^3}{6} + \frac{1}{2x}).\frac{1}{3}(6 + 2x^2)^{-\frac{2}{3}} \cdot (6 + 2x^2)' =$

$= (\frac{x^2}{2} - \frac{1}{2x^2}).(6 + 2x^2)^{\frac{1}{3}} + (\frac{x^3}{6} + \frac{1}{2x}).(6 + 2x^2)^{\frac{-2}{3}} \cdot \frac{4x}{3}$

then

$f'(-1) = 0 + \left(\frac{-1}{6} - \frac{1}{2}\right)\left(\frac{-4}{3}\right)8^{-\frac{2}{3}} = \frac{8}{9}\left(\sqrt[3]{8}\right)^{-2} = \frac{2}{9}.$

For 6)

by Quotient Rule

$$y' = \frac{(2x)'(1+3x) - 2x(1+3x)'}{(1+3x)^2} = \frac{2(1+3x) - 2x\cdot 3}{(1+3x)^2} = \frac{2}{(1+3x)^2}$$

the slope of a parallel tangent $\quad y' = 2 \quad (y = 2x+3)$

solve for $x \quad \dfrac{2}{(1+3x)^2} = 2 \quad (1+3x)^2 = 1 \quad 1+3x = \pm 1$

OR $\quad 1 + 6x + 9x^2 = 1 \quad 3x(2+3x) = 0$

points are $x = 0, y = 0$ and $x = -\frac{2}{3}, y = \left.\dfrac{2x}{1+3x}\right|_{x=-\frac{2}{3}} = \frac{\frac{-4}{3}}{1-2} = \dfrac{4}{3}$

$(0,0)$ and $\left(-\dfrac{2}{3}, \dfrac{4}{3}\right)$.

For 7)

$$f'(\tfrac{1}{2}) = \lim_{x\to\frac{1}{2}} \frac{f(x) - f(\frac{1}{2})}{x - \frac{1}{2}} = \lim_{x\to\frac{1}{2}} \frac{2x - \frac{1}{x} - (1-2)}{x - \frac{1}{2}} = \lim_{x\to\frac{1}{2}} \frac{2x - \frac{1}{x} + 1}{x - \frac{1}{2}} = \lim_{x\to\frac{1}{2}} \frac{\frac{2x^2 - 1 + x}{x}}{\frac{2x-1}{2}} =$$

$$= \lim_{x\to\frac{1}{2}} \frac{2(2x^2 + x - 1)}{x(2x-1)} = \lim_{x\to\frac{1}{2}} \frac{2(2x-1)(x+1)}{x(2x-1)} = \lim_{x\to\frac{1}{2}} \frac{2(x+1)}{x} = 4\cdot\frac{3}{2} = 6$$

OR

$$f'(\tfrac{1}{2}) = \lim_{h\to 0} \frac{f(\frac{1}{2}+h) - f(\frac{1}{2})}{h} = \lim_{h\to 0} \frac{2(\frac{1}{2}+h) - \frac{1}{\frac{1}{2}+h} - (1-2)}{h} =$$

$$= \lim_{h\to 0} \frac{1 + 2h - \frac{2}{1+2h} + 1}{h} = \lim_{h\to 0} \frac{1}{h}\left[\frac{2(1+h)(1+2h) - 2}{1+2h}\right] =$$

$$= \lim_{h\to 0} \frac{2}{h}\left[\frac{1 + 3h + 2h^2 - 1}{1+2h}\right] = \lim_{h\to 0} 2\left[\frac{3+2h}{1+2h}\right] = 6$$

check by Rules $\quad f'(x) = 2 + \frac{1}{x^2}$ so at $x = \frac{1}{2}$ we get 6.

For 8)

$$y = \sqrt{7x + \frac{3}{x^2} + 4\sqrt{x}} = \left(7x + \frac{3}{x^2} + 4\sqrt{x}\right)^{\frac{1}{2}} \text{ by Chain Rule}$$

$$y' = \tfrac{1}{2}\left(7x + \frac{3}{x^2} + 4\sqrt{x}\right)^{-\frac{1}{2}} \left(7x + 3x^{-2} + 4x^{\frac{1}{2}}\right)' = \frac{7 - 6x^{-3} + 4\cdot\frac{1}{2}x^{-\frac{1}{2}}}{2\sqrt{7x + \frac{3}{x^2} + 4\sqrt{x}}}$$

$$y' = \frac{7 - \frac{6}{x^3} + \frac{2}{\sqrt{x}}}{2\sqrt{7x + \frac{3}{x^2} + 4\sqrt{x}}} \text{ for } x > 0.$$

For 9)

by Quotient Rule $\quad y' = \dfrac{(2x-3)'(4-2x^5) - (2x-3)(4-2x^5)'}{(4-2x^5)^2}$

so $\quad y' = \dfrac{(2)(4-2x^5) - (2x-3)(-10x^4)}{(4-2x^5)^2}$ at $x = -1$

slope $\quad m = f'(-1) = \dfrac{2\cdot 6 - (-5)(-10)}{36} = \dfrac{-38}{36} = \dfrac{-19}{18}$

and $f(-1) = \frac{-5}{6} \quad P\left(-1, -\dfrac{5}{6}\right)$

so tangent $\quad y = \frac{-19}{18}(x+1) - \frac{5}{6}$ or $y = \frac{-19}{18}x - \frac{34}{18}$ or $18y + 19x = -34$.